Prof. Dr. J. Giesl F. Frohn

Exam in Functional Programming SS 2014 (V3M)

First Name: ______________________________

Last Name: ______________________________

Matriculation Number: ______________________________

Course of Studies (please mark **exactly** one):

- ○ Informatik Bachelor
- ○ Informatik Master
- ○ Other: ______________________
- ○ Mathematik Master
- ○ Software Systems Engineering Master

	Available Points	Achieved Points
Exercise 1	**9**	
Exercise 2	**29**	
Exercise 3	**24**	
Exercise 4	**18**	
Exercise 5	**10**	
Sum	**90**	

Notes:

- **On all sheets** (including additional sheets) you must write **your first name, your last name and your matriculation number**.
- Give your answers in readable and understandable form.
- Use **permanent** pens. Do not use red or green pens and do not use pencils.
- Please write your answers on the exam sheets (also use the reverse sides).
- For each exercise part, give **at most one** solution. Cancel out everything else. Otherwise all solutions of the particular exercise part will give you **0 points**.
- If we observe any **attempt of deception**, the whole exam will be evaluated to **0 points**.
- At the end of the exam, hand in **all sheets together with the sheets containing the exam questions**.

Exercise 1 (Quiz): (3 + 3 + 3 = 9 points)

a) Give a type declaration for `f` such that `(f True) (f 1)` is well typed in Haskell or explain why such a type declaration cannot exist.

b) Prove or disprove: If $\succ \subseteq A \times A$ is confluent, then each $a \in A$ has at most one normal form w.r.t. $\succ$.

c) What is the connection between monotonicity, continuity, and computability?

Exercise 2 (Programming in Haskell): (6 + 7 + 7 + 9 = 29 points)

We define a polymorphic data structure `Tree e` for binary trees whose nodes store values of type `e`.

```
data Tree e = Node e (Tree e) (Tree e) | Empty
```

The data structure `Forest e` is used to represent lists of trees.

```
type Forest e = [Tree e]
```

Furthermore, we define the following data structure:

```
data Animal = Squirrel | None
```

For example, `aForest` is a valid expression of type `Forest Animal`.

```
aForest = [Node Squirrel Empty (Node Squirrel Empty Empty), Node None Empty Empty]
```

In this exercise, you may use full Haskell and predefined functions from the Haskell Prelude.

a) Implement a function `hunt` together with its type declaration that removes all `Squirrel`s from a `Forest Animal`, i.e., each occurrence of a `Squirrel` should be replaced by `None`.

For example, `hunt aForest` should be evaluated to `[Node None Empty (Node None Empty Empty), Node None Empty Empty]`.

b) Implement a function `fold :: (e -> res -> res -> res) -> res -> Tree e -> res` to fold a `Tree`. The first argument of `fold` is the function that is used to combine the value of the current `Node` with the subresults obtained for the two direct subtrees of the current `Node`. The second argument of `fold` is the start value, i.e., the initial subresult. The third argument is the `Tree` that has to be folded. So for a `Tree t`, `fold f x t` replaces the constructor `Node` by `f` and the constructor `Empty` by `x`.

As an example, consider the following function:

```
count :: Animal -> Int -> Int -> Int
count Squirrel x y = x + y + 1
count None x y = x + y
```

Then `fold count 0 (Node Squirrel Empty (Node Squirrel Empty Empty))` should evaluate to `2`, i.e., this application of `fold` counts all `Squirrels` in a `Tree`.

c) Implement a function `isInhabited` together with its type declaration which gets a `Forest Animal` as input and returns `True` if and only if there is a `Tree` in the `Forest` that contains a `Squirrel`. For the definition of `isInhabited`, use only one defining equation where the right-hand side contains a call to the function `fold`. Of course, you may (and have to) use the function `fold` even if you were not able to solve exercise part (b). Moreover, you may use the function `count` from exercise part (b).

Note that the function `fold` operates on a `Tree`, whereas the function `isInhabited` operates on a `Forest`!

d) In this exercise, you should implement a game where the user controls a lumberjack (i.e., a person working in a forest). The lumberjack walks through a `Forest Animal` and wants to cut down all `Trees` with as few moves as possible without damaging `Squirrels`. A move is either cutting down the current `Tree` or rescuing all `Squirrels` from the current `Tree` (such that it can be cut down safely, afterwards).

Implement a function `lumberjack :: Forest Animal -> IO ()` that works as follows:

It starts at the first `Tree` of the forest and prints `"What do you want to do? (cut down (c), rescue squirrels (r))"`. If the user answers `"r"`, then all `Squirrels` are removed from the `Tree` and the user is asked to choose an action again. This also happens if the `Tree` does not contain any `Squirrels`. If the user answers `"c"` and the `Tree` contained `Squirrels`, then the function prints `"You cut down a tree with squirrels!"` and terminates. If the user answers `"c"` and the `Tree` does not contain `Squirrels`, then the function continues with the next `Tree`, if any, and the user is asked to choose an action again. If the user's answer is neither `"r"` nor `"c"`, then the function asks to choose an action again. If no `Trees` are left, the function prints `"You cut down all trees with` n `moves!"` (where n is the number of moves that were performed) and terminates.

You can assume that there exists a function `searchSquirrels :: Tree Animal -> Bool` which checks whether there are `Squirrels` in a `Tree` and a function `rescue :: Tree Animal -> Tree Animal` that replaces all occurrences of `Squirrel` in the given `Tree` with `None`. So, for example, `searchSquirrels (Node None Empty (Node Squirrel Empty Empty))` evaluates to `True` and `rescue (Node None Empty (Node Squirrel Empty Empty))` evaluates to `Node None Empty (Node None Empty Empty)`.

A successful run of `lumberjack` could look as follows:

```
*Main> lumberjack aForest
What do you want to do? (cut down (c), rescue squirrels (r)) r
What do you want to do? (cut down (c), rescue squirrels (r)) c
What do you want to do? (cut down (c), rescue squirrels (r)) c
You cut down all trees with 3 moves!
```

In the following run, the user looses the game:

```
*Main> lumberjack aForest
What do you want to do? (cut down (c), rescue squirrels (r)) c
You cut down a tree with squirrels!
```

Hint: You should use the function `getLine :: IO String` to read the input from the user. To print a `String`, you should use the function `putStr :: String -> IO ()` or the function `putStrLn :: String -> IO ()`, if the output should end with a line break. You should use the function `show :: Int -> String` to convert an `Int` to a `String`. To save space, you may assume that the following declarations exist in your program:

```
chooseAction, lost :: String
chooseAction = "What do you want to do? (cut down (c), rescue squirrels (r)) "
lost = "You cut down a tree with squirrels!"
```

Exercise 3 (Semantics): (12 + 7 + 5 = 24 points)

a) i) Let $\mathbb{N}^\infty$ be the set of all infinite sequences of natural numbers (e.g., $[0, 0, 2, 2, 4, 4, \dots] \in \mathbb{N}^\infty$) and let $\leq_p \subseteq \mathbb{N}^\infty \times \mathbb{N}^\infty$ be the relation that compares infinite sequences of natural numbers by their *prefix sums*. The n^{th} prefix sum $p_n(s)$ for some $n \in \mathbb{N}$ of a sequence $s \in \mathbb{N}^\infty$ is the sum of the first n elements of s. We have $s \leq_p s'$ if and only if $s = s'$ or there is an $n \in \mathbb{N}$ such that $p_n(s) < p_n(s')$ and $p_m(s) = p_m(s')$ for all $m \in \{0, \dots, n-1\}$.

1) Prove that $\leq_p$ is transitive.

2) Give an example for an infinite chain in $(\mathbb{N}^\infty, \leq_p)$.

3) Prove or disprove: The partial order $\leq_p$ is complete on $\mathbb{N}^\infty$.

ii) Prove or disprove: The partial order $\leq$ is complete on $\mathbb{N}$. Here, $\leq$ is the usual "less than or equal" relation.

b) i) Consider the following Haskell function `f`:

```
f :: Int -> Int
f 0 = 1
f x = x * x * f (x - 1)
```

Please give the Haskell declaration for the higher-order function `ff` corresponding to `f`, i.e., the higher-order function `ff` such that the least fixpoint of `ff` is `f`. In addition to the function declaration, please also give the type declaration of `ff`. You may use full Haskell for `ff`.

ii) Let $\phi_{\texttt{ff}}$ be the semantics of the function `ff`. Give the least fixpoint of $\phi_{\texttt{ff}}$ in closed form, i.e., give a non-recursive definition of the least fixpoint of $\phi_{\texttt{ff}}$.

Hint: For natural numbers x, the factorial function can be defined as follows:

$$x! = \begin{cases} 1 & \text{if } x = 0 \\ x \cdot (x-1)! & \text{if } x > 0 \end{cases}$$

c) Consider the data type declarations on the left and, as an example, the graphical representation of the first three levels of the domain for `Nats` on the right:

```
data Nats = Z | S Nats

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type Forest e = [Tree e]
data Tree e =
    Node e (Tree e) (Tree e)
    | Empty
```

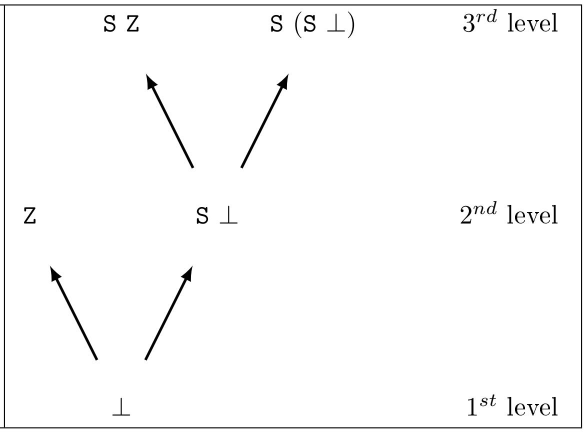

Give a graphical representation of the first three levels of the domain for the type `Forest Int`. The third level contains the element `Empty:`$\perp$, for example.

Exercise 4 (Lambda Calculus): (4 + 8 + 6 = 18 points)

a) Reconsider the function `f` from the previous exercise:

```
f :: Int -> Int
f 0 = 1
f x = x * x * f (x - 1)
```

Please implement this function in the Lambda Calculus, i.e., give a term `t` such that, for all `x`, `y` $\in \mathbb{Z}$, `f x == y` if and only if `t x` can be reduced to `y` via WHNO-reduction with the $\rightarrow_{\beta\delta}$-relation and the set of rules δ as introduced in the lecture to implement Haskell. You can use infix notation for predefined functions like $(==)$, $(*)$ or $(-)$.

b) Let $\mathtt{t} = \lambda g\ x.\mathtt{if}\ (x == 0)\ x\ (g\ x)$ and

$$\begin{aligned}\delta = \ & \{\ \mathtt{if\ True} \rightarrow \lambda\, x\ y.\, x,\\ & \quad \mathtt{if\ False} \rightarrow \lambda\, x\ y.\, y,\\ & \quad \mathtt{fix} \rightarrow \lambda\, f.\, f(\mathtt{fix}\ f)\}\\ \cup & \ \{\ x == x \rightarrow \mathtt{True} \mid x \in \mathbb{Z}\}\\ \cup & \ \{\ x == y \rightarrow \mathtt{False} \mid x, y \in \mathbb{Z}, x \neq y\}\end{aligned}$$

Please reduce $\mathtt{fix}\,\mathtt{t}\,0$ by WHNO-reduction with the $\rightarrow_{\beta\delta}$-relation. List **all** intermediate steps until reaching weak head normal form, but please write "`t`" instead of $\lambda g\ x.\mathtt{if}\ (x == 0)\ x\ (g\ x)$ whenever possible.

c) Consider the Boolean operator `nand` where `nand`(x, y) holds if and only if `and`(x, y) does *not* hold. Using the representation of Boolean values in the pure λ-calculus presented in the lecture, i.e., `True` is represented as $\lambda\, x\ y.\, x$ and `False` as $\lambda\, x\ y.\, y$, give a pure λ-term for `nand` in $\to_\beta$-normal form.

In your solution, you may abbreviate the λ-term $\lambda\, x\ y.\, x$ with `True` and the λ-term $\lambda\, x\ y.\, y$ with `False`.

Exercise 5 (Type Inference): (10 points)

Using the initial type assumption $A_0 := \{x :: \forall a.a, g :: \forall a.a\}$, infer the type of the expression $\lambda f.g\,(f\,x)$ using the algorithm $\mathcal{W}$.