Stochastik, 1. Übung, 26.4.01

EidS Übungen 2001

Aufgabe 1:

a) $\Omega_I = \{ (\omega_1, \omega_2, \omega_3, \omega_4, \ldots, \omega_{52}) \in A^{n} \mid \omega_i \neq \omega_j,\ i \neq j \ \ 1 \leq i,j \leq n \}$

mit $A = \{1, \ldots, 52\}$ Karten 1-13 Herz, 14-26 Karo, ⋮

$A = \{ (\omega_1, \omega_2, \ldots, \omega_{52}) \in \Omega_I \mid \omega_1 < \ldots < \omega_{13} \leq 13 < \omega_{14} < \ldots < \omega_{26} \leq 26 \leq \ldots \}$

$|A| = 13! \cdot 13! \cdot 13! \cdot 13! = (13!)^4$

$\Rightarrow P(A) = \frac{|A|}{|\Omega_I|} = \frac{(13!)^4}{52!} \approx 1{,}86 \cdot 10^{-29}$

b) $\Omega_{II} = \{ (\omega_1, \ldots, \omega_{13}) \in A^{13} \mid \omega_1 < \omega_2 < \ldots < \omega_{13} \}$

Asse haben Nr. 1..4

$|\Omega_{II}| = \binom{N}{n} = \binom{52}{13}$

$A = \{ (\omega_1, \omega_2, \ldots, \omega_{13}) \in \Omega_{II} \mid \omega_1 \in \{1,2,3,4\},\ \omega_2, \ldots, \omega_{13} \notin \{1,2,3,4\} \}$

$|A| = \binom{4}{1}\binom{48}{12}$

$P(A) = \frac{\binom{4}{1}\binom{48}{12}}{\binom{52}{13}} \approx 0{,}44$

c) schwarze Karten: $S = \{1, \ldots, 26\}$
rote " : $R = \{27, \ldots, 52\}$

$A_i = \{ (\omega_1, \ldots, \omega_{13}) \in \Omega_{II} \mid \omega_k \leq 26 \ \ \forall k \in \{1, 2, \ldots, 13\},\ k \leq i;\ \omega_j > 26 \ \ \forall j > i \}$

$\hat{=}$ genau i schwarze Karten

$|A_i| = \binom{26}{i}\binom{26}{13-i}$

$A = A_0 \cup A_1 \cup \ldots \cup A_4$

$|A| = \sum_{i=0}^{4} |A_i| = \sum_{i=0}^{4} \binom{26}{i}\binom{26}{13-i}$

$P(A) = \frac{|A|}{|\Omega_{II}|} \approx 0{,}10$

Behauptung: $|A_{k,m}| = \begin{cases} n^m \binom{m}{k} \left(\frac{s}{n}\right)^k \left(1-\frac{s}{n}\right)^{m-k} & k \le m \\ 0 & \text{sonst} \end{cases}$

Bew. per Ind.: (nach m)

IA: $m=1$: $A_{0,1} = \{s+1, \dots, n\}$ $\quad A_{1,1} = \{1, \dots, s\}$ $\quad A_{k,1} = \emptyset$ für $k \notin \{0,1\}$

$$|A_{k,1}| = \begin{cases} n-s & k=0 \\ s & k=1 \\ 0 & \text{sonst} \end{cases}$$

$m \to m+1$: $n-s$ Mögl. im $(m+1)$-ten Versuch keinen mark. Fisch zu fangen
s —— " —— einen mark. Fisch zu fangen

$$\begin{aligned}
|A_{k,m+1}| &= (n-s)\,|A_{k,m}| + s\,|A_{k-1,m}| \\
&\overset{IV}{=} (n-s)\, n^m \binom{m}{k} \left(\frac{s}{n}\right)^k \left(1-\frac{s}{n}\right)^{m-k} + s \cdot n^m \binom{m}{k-1} \left(\frac{s}{n}\right)^{k-1} \left(1-\frac{s}{n}\right)^{m-k+1} \\
&= n^m \left(\frac{s}{n}\right)^{k-1} \left(1-\frac{s}{n}\right)^{m-k} \left[(n-s)\binom{m}{k}\left(\frac{s}{n}\right) + s\binom{m}{k-1}\left(1-\frac{s}{n}\right)\right] \\
&= s\, n^m \left(\frac{s}{n}\right)^{k-1} \left(1-\frac{s}{n}\right)^{m-k+1} \left[\binom{m}{k} + \binom{m}{k-1}\right] \\
&= n^{m+1} \left(\frac{s}{n}\right)^k \left(1-\frac{s}{n}\right)^{m-k+1} \binom{m+1}{k} \quad \checkmark
\end{aligned}$$

Gleichverteilung

Aufgabe 2.7:

$A, B \in \mathfrak{A}$, W-raum $(\Omega, \mathfrak{A}, P)$

z.z.: $|P(A) - P(B)| \leq P(A \cap B^c) + P(A^c \cap B)$

dann: $P(A \cap B^c) + P(A^c \cap B) = 1 - P((A \cap B^c)^c) + P(A^c \cap B)$

$= 1 - P(A^c \cup B) + P(A^c \cap B)$

$\geq 1 - P(A^c) - P(B) = P(A) - P(B)$

$P(A \cap B^c) + P(A^c \cap B) = P(A \cap B^c) + 1 - P(A \cup B^c)$

$\geq 1 - P(A) - P(B^c) = P(B) - P(A)$

$\Rightarrow \quad \ldots \geq |P(A) - P(B)|$

Aufgabe 4:

a) bez. x die Ankunftszeit von A, y die Ankunftszeit von B.

Wartezeit: $z = |x - y|$

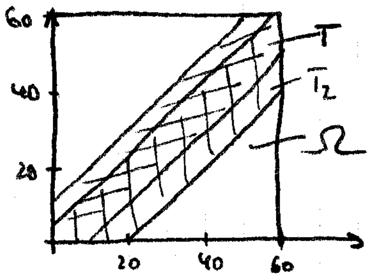
Damit sie sich treffen: $z = |x - y| \leq 10$

$\Omega = \{ (x, y) \in \mathbb{R}^2 \mid 0 \leq x, y \leq 60 \}$

$T = \{ (x, y) \in \Omega \mid |x - y| \leq 10 \}$

$|x - y| \leq 10 \iff -10 \leq x - y \leq 10$

$\iff (y \geq x - 10 \wedge y \leq x + 10)$



$\mu(\Omega) = 60^2$

$\mu(T) = 60^2 - 2\left(\frac{1}{2} 50^2\right)$

$\Rightarrow P(T) = \frac{\mu(T)}{\mu(\Omega)} = \frac{60^2 - 50^2}{60^2} = 1 - \left(\frac{5}{6}\right)^2 = \frac{11}{36} \approx 0{,}306$

b) $T_2 = \{ (x, y) \mid x \leq y + 20 \wedge y \leq x + 5 \} = \{ (x, y) \in \Omega \mid x - 20 \leq y \leq x + 5 \}$

$\mu(T_2) = 60^2 - \left(\frac{1}{2} 40^2 + \frac{1}{2} 55^2\right)$

$\Rightarrow P(T_2) = \frac{\mu(T_2)}{\mu(\Omega)} = \frac{60^2 - \left(\frac{1}{2} 40^2 + \frac{1}{2} 55^2\right)}{60^2} = \frac{103}{288} \approx 0{,}358$

Aufgabe 6:

$(\Omega, \mathfrak{A}, P)$, $A, B \in \mathfrak{A}$

a) $P(A^c) = 1 - P(A)$

dazu: $1 \underset{\text{Def. 2.10(i)}}{=} P(\Omega) = P(A \dot\cup A^c) \overset{(*)}{\underset{\text{Def. 2.10(ii)}}{=}} P(A) + P(A^c)$

Bem.: $(*)$ folgt aus $P(\dot\bigcup_{n\in\mathbb{N}} A_n) = \sum_{n=1}^{\infty} P(A_n)$ mit $A_n = \emptyset$ für alle $n > m$

d) $P(B) - P(A) = P(B \setminus A)$ für $A \subset B$

dazu: $P(B) \underset{A\subset B}{=} P(A \dot\cup (B\setminus A)) = P(A) + P(B \setminus A)$

b) z.z. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

dazu: $P(A \cup B) = P(A \dot\cup (B \setminus A)) = P(A) + P(B \setminus A) = P(A) + P(B \setminus (A \cap B))$

$= P(A) + P(B) - P(A \cap B)$

c) z.z.: $P(A) \leq P(B)$ für $A \subset B$

dazu: $P(B \setminus A) \underset{d)}{=} P(B) - P(A) \geq 0 \quad \Rightarrow \quad P(B) \geq P(A)$

e) z.z.: $\{A_n\}$ absteigend $\Rightarrow P(\lim_{n\to\infty} (A_n)) = \lim_{n\to\infty} P(A_n)$

dazu: $\{A_n\}$ absteigend $\rightarrow \{A_n^c\}$ aufsteigend

$\Rightarrow P(\bigcap_{n=1}^{\infty} A_n) = 1 - P(\bigcup_{n=1}^{\infty} A_n^c) \underset{\text{Lemma 2.11 e)1}}{=} 1 - \lim_{n\to\infty} P(A_n^c) = \lim_{n\to\infty} P(A_n)$

Aufgabe 5:

a) $\Omega, I \neq \emptyset$ $\quad \{\mathfrak{A}_i\}_{i\in I}$ Familie von σ-Algebren über Ω

z.z. $\mathfrak{A} := \bigcap_{i\in I} \mathfrak{A}_i$ ist σ-Algebra

dazu: (i) z.z. $\Omega \in \mathfrak{A} = \bigcap_{i\in I} \mathfrak{A}_i$

$\forall i \in I: \Omega \in \mathfrak{A}_i \Rightarrow \Omega \in \bigcap_{i\in I} \mathfrak{A}_i = \mathfrak{A}$

(ii) $A \in \bigcap_{i\in I} \mathfrak{A}_i \Rightarrow A^c \in \bigcap_{i\in I} \mathfrak{A}_i$

$\forall i \in I: A \in \mathfrak{A}_i \Rightarrow A^c \in \mathfrak{A}_i \Rightarrow A^c \in \bigcap_{i\in I} \mathfrak{A}_i = \mathfrak{A}$

(iii) z.z.: $A_k \in \mathfrak{A}$, $k \in \mathbb{N} \Rightarrow \bigcup_{k=1}^{\infty} A_k \in \mathfrak{A}$

$\forall k \in \mathbb{N}, \forall i \in I: \quad A_k \in \mathfrak{A} \Rightarrow A_k \in \mathfrak{A}_i \Rightarrow \bigcup_{k=1}^{\infty} A_k \in \mathfrak{A}_i \Rightarrow \bigcup A_k \in \mathfrak{A} = \bigcap_{i\in I} \mathfrak{A}_i$

b) Gegbsp.: $I = \{1,2\}$, $\Omega = \{1,2,3\}$; $\mathfrak{A}_1 = \{\emptyset, \{1\}, \{2,3\}, \Omega\}$ $\quad \mathfrak{A}_2 = \{\emptyset, \{2\}, \{1,3\}, \Omega\}$; $\mathfrak{A}_1 \cup \mathfrak{A}_2$ keine σ-Algebra

Aufgabe 10:

a) naiver Ansatz: $\Omega = \{a, b, c\}$, $a \in \mathcal{P}(\Omega)$ $\quad P(\{\omega\}) = \frac{1}{3}$

Interpretation: $\{\omega\}$ heißt: ω wird begnadigt.

$$P(\{a\}) = \frac{1}{3}$$

Wärter nennt $B \Rightarrow B = \{a, c\}$

$$P(\{a\} \mid B) = \frac{P(\{a\})}{P(\{a,c\})} = \frac{1}{2}$$

b) besseres Modell:

$$\Omega = \{(a,b), (a,c), (b,c), (c,b)\}$$

Interpretation: $\omega = (\omega_1, \omega_2)$ heißt: ω_1 wird begnadigt, ω_2 wird vom Wärter genannt

$$P(\{(b,c)\}) = P(\{c,b\}) = P(\{(a,b),(a,c)\}) = \frac{1}{3}$$

$$P(\{(a,b),(a,c)\}) = \underbrace{P(\{(a,b)\}) + P(\{(a,c)\})}_{\text{Verhältnis muss bekannt sein!}} = \frac{1}{3}$$

c) Es sei $P(\{(a,b)\}) = \frac{p}{3} \Rightarrow P(\{(a,c)\}) = \frac{1-p}{3}$

$A \mathrel{\hat{=}} A$ wird begnadigt: $A = \{(a,b), (a,c)\}$

$B \mathrel{\hat{=}}$ Wärter nennt B: $B = \{(a,b), (c,b)\}$

$$P(A \mid B) = \frac{P(\{a,b\})}{P(\{(a,b),(c,b)\})} = \frac{\frac{p}{3}}{\frac{p}{3} + \frac{1}{3}} = \frac{p}{1+p}$$

1) $p = \frac{1}{2} \Rightarrow P(A \mid B) = \frac{1}{3}$

2) $p \geq \frac{1}{2} \Rightarrow P(A \mid B) > \frac{1}{3}$

Aufgabe 8:

a) $A \hat{=}$ Prozessor wird von Kontrolle als Ausschuss deklariert

$F \hat{=}$ Prozessor ist fehlerhaft

geg.: $P(A) = 0{,}1$ gesucht (eigentlich): $P(F)$, $P(F|A)$, $P(F|\bar{A})$

$P(A|F) = 0{,}94$

$P(A|F^C) = 0{,}042$

i) $P(A) = P(A|F) \cdot P(F) + P(A|\bar{F}) \cdot P(\bar{F})$ (Satz v.d. tot. W'keit)

$= P(A|F) \cdot P(F) + P(A|\bar{F}) \cdot (1 - P(F))$

$= P(A|\bar{F}) + (P(A|F) - P(A|\bar{F})) \cdot P(F)$

$\Rightarrow P(F) = \dfrac{P(A) - P(A|\bar{F})}{P(A|F) - P(A|\bar{F})} = 0{,}065$

ii) $P(F|A) \hat{=}$ W'keit, dass Prozessor defekt und als Ausschuss deklariert

$$P(F|A) \underset{\text{Bayes}}{=} \frac{P(A|F) \cdot P(F)}{P(A|F) \cdot P(F) + P(A|\bar{F}) \cdot P(\bar{F})} = \frac{P(A|F) \cdot P(F)}{P(A)}$$

iii) $P(F|\bar{A}) = \dfrac{P(\bar{A}|F) \cdot P(F)}{P(\bar{A})} = 0{,}0043$

Aufgabe 9:

Ereignisse $E_i := \{1, i\} \in \mathcal{P}(\{1,2,3,4\})$; $i \in \{2,3,4\}$

$P(\{i\}) = \frac{1}{4}$

Es gilt $\forall i \neq j \in \{2,3,4\}$: $P(E_i \cap E_j) = P(\{1\}) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = P(E_i) \cdot P(E_j)$

$\Rightarrow$ paarw. stoch. unabhg.

$P(E_2 \cap E_3 \cap E_4) = P(\{1\}) = \frac{1}{4} \neq \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = P(E_2) \cdot P(E_3) \cdot P(E_4)$

$\Rightarrow E_2, E_3, E_4$ nicht (gem.) s.u.

Aufgabe 8:

b) A_1, A_2 : Ausschuss in 1./2. Kontrolle

$$P(A_1) = P(A_2) = 0{,}1$$

$$P(A_1|F) = P(A_2|F) = 0{,}94$$

$$P(A_1|\bar F) = P(A_2|\bar F) = 0{,}042$$

$$P(F) = \frac{58}{898} = 0{,}065 \quad \text{(s. a))}$$

gesucht: $P(F|A_1 \cap A_2)$ und $P(F|\overline{A_1 \cap A_2})$

$$P(F|A_1\cap A_2) = \frac{P(A_1\cap A_2\cap F)}{P(A_1\cap A_2)} = \frac{P(A_1\cap A_2|F)\cdot P(F)}{P(A_1\cap A_2)} \leftarrow P(A_1\cap A_2|F)\,P(F) + P(A_1\cap A_2|\bar F)\,P(\bar F)$$

$$P(A_1\cap A_2|F) = \frac{P(A_1\cap A_2\cap F)}{P(F)} = \frac{P(A_2|A_1\cap F)\,P(A_1\cap F)}{P(F)}$$

$$\overset{*}{=} \frac{P(A_2|F)\cdot P(A_1|F)\cdot P(F)}{P(F)}$$

$$= P(A_2|F)\cdot P(A_1|F) = 0{,}94^2$$

(*) gleich, da die beiden Kontrollen voneinander unabhängig sind

$$\cancel{P(A_2|A_1\cap F) = \frac{P(A_1\cap F\cap A_2)}{P(A_1\cap F)}}$$

$$P(A_1\cap A_2) = P(A_1\cap A_2|F)\,P(F) + P(A_1\cap A_2|\bar F)\,P(\bar F)$$

$$= P(A_1|F)\,P(A_2|F)\,P(F) + P(A_1|\bar F)\,P(A_2|\bar F)\cdot P(\bar F) \approx 0{,}0587$$

$$\Rightarrow P(F|A_1\cap A_2) \approx 0{,}9719$$

$$P(F|\overline{A_1\cap A_2}) = \frac{P(F\cap(\overline{A_1\cap A_2}))}{P(\overline{A_1\cap A_2})} = \frac{P(F\cap(\bar A_1 \cup \bar A_2))}{1-P(A_1\cap A_2)} =$$

$$= \frac{P(F\cap((\bar A_1\cap\bar A_2)\,\dot\cup\,(\bar A_1\cap A_2)\,\dot\cup\,(A_1\cap\bar A_2)))}{1-P(A_1\cap A_2)}$$

$$= \frac{P(F\cap\bar A_1\cap\bar A_2) + P(F\cap\bar A_1\cap A_2) + P(F\cap A_1\cap\bar A_2)}{1-P(A_1\cap A_2)}$$

$$= \frac{P(F)\,\left(P(\bar A_1|F)\,P(\bar A_2|F) + P(\bar A|F)\,P(A_2|F) + P(A_1|F)\cdot P(\bar A_2|F)\right)}{1-P(A_1\cap A_2)}$$

$$\approx 0{,}008$$

Aufgabe 11:

A_i : Router R_i intakt $\overset{(s.u.)}{,}$ $P(A_i) = p_i$ $i=1,\dots,4$

I : Verbindung $A \to B$ möglich

$$P(I) = \underbrace{P(\cancel{A}\, I \mid A_1)}_{P(A_3 \cup A_4)} P(A_1) + \underbrace{P(I \mid A_1)}_{P((A_2 \cap A_3) \cup (A_2 \cap A_4)) = P(A_2 \cap A_3) + P(A_2 \cap A_4) - P(A_2 \cap A_3 \cap A_4)} P(\overline{A_1})$$

$$P(A_3 \cup A_4) = P(\overline{\overline{A_3} \cap \overline{A_4}}) = 1-(1-p_3)(1-p_4)$$

$$\Rightarrow P(I) = p_1 \cdot (1-(1-p_3)(1-p_4)) + (p_2 p_3 + p_2 p_4 - p_2 p_3 p_4)(1-p_1)$$

$$= p_1 p_3 + p_1 p_4 - p_1 p_3 p_4 + p_2 p_3 + p_2 p_4 - p_2 p_3 p_4 - p_1 p_2 p_4 - p_1 p_2 p_3 + p_1 p_2 p_3 p_4$$

Übung 3 (AN IV) 9.5.01

Ziegenproblem:

A_i : Auto hinter Tür i

M_i : Moderator öffnet Tür i

<u>Vor.</u>: Kandidat hat Tür 1 gewählt!

$P(A_1 | M_3)$ vs. $P(A_2 | M_3)$

$$P(A_1|M_3) = \frac{P(M_3|A_1)\cdot P(A_1)}{\sum_{i=1}^{3} P(M_3|A_i)\cdot P(A_i)} \qquad P(A_2|M_3) = \frac{P(M_3|A_2)\cdot P(A_2)}{\sum_{i=1}^{3} P(M_3|A_i)\cdot P(A_i)}$$

$P(A_1) = P(A_2) = P(A_3) = \frac{1}{3}$

$P(M_3 | A_3) = 0$

$P(M_3 | A_2) = 1$

$P(M_3 | A_1) = \alpha$

also: $$P(A_1|M_3) = \frac{\frac{1}{3}\alpha}{\left(\frac{1}{3}\alpha + \frac{1}{3}\right)} = \frac{\alpha}{\alpha+1}$$

$$P(A_2|M_3) = \frac{\frac{1}{3}}{\frac{1}{3}\alpha + \frac{1}{3}} = \frac{1}{\alpha+1}$$

Wenn Moderator "zufällig" auswählt : $\alpha = \frac{1}{2}$

$$P(A_1|M_3) = \frac{\frac{1}{2}}{\frac{1}{2}+1} = \frac{1}{3}$$

$$P(A_2|M_3) = \frac{1}{\frac{1}{2}+1} = \frac{2}{3}$$

zu Aufgabe 3.2:

$\Omega = \{ ①, ②, ③, ④ \} \cong \{1,2,3,4\}$

Elementarereignis: $\overset{\text{genau}}{1}$ Ball wird gezogen (+ es gibt nur eine Möglichkeit)

$\{1,3\}$ ist kein Elementarereignis (Ball mit ungerader Nr. ziehen)

$P(\{i\}) = \frac{1}{4} \qquad i = 1, \dots, 4$

ges.: $A, B, C \subseteq \Omega$ mit
$$P(A \cap B) = P(A) \cdot P(B)$$
$$P(B \cap C) = P(B) \cdot P(C)$$
$$P(A \cap C) = P(A) \cdot P(C)$$
(stoch. unabhgg.)

aber nicht gemeinsam stoch. unabhg.

also $P(A \cap B \cap C) \neq P(A) \cdot P(B) \cdot P(C)$

wg. Laplace (alle Ereignisse gleichwahrscheinlich): $P(A) = |A|$

zu Aufgabe 3.1:

a) A: Ausschuss $\qquad P(A) = 0{,}1$

D: defekt $\qquad P(A|D^c) = 0{,}042$

$P(A|D) = 0{,}94$

$P(D)$?

b) zwei Prüfungen, stoch. unabhg. $\qquad A'$: Ausschuss nach 2 Überprüfungen

$P(A'|D^c) = P(A|D^c)^2$

$P(A'|D) = P(A|D)^2$

$P(A')$?

Aufgabe 14:

a) "$\Rightarrow$" X ist ZV $\Rightarrow \forall B \in \mathcal{B}^1: X^{-1}(B) \in \mathfrak{A} \Rightarrow \forall \alpha \in \mathbb{R}: \{X \le \alpha\} = X^{-1}((-\infty,\alpha]) \in \mathfrak{A}$

$(-\infty,\alpha] \in \mathcal{B}^1$

Zeige Hinweis: $\mathfrak{X} = \{B \subseteq \mathbb{R} \mid X^{-1}(B) \in \mathfrak{A}\}$. Zeige $\mathfrak{X}$ ist σ-Algebra über $\mathbb{R}$

i) $X^{-1}(\mathbb{R}) = \Omega \in \mathfrak{A} \Rightarrow \mathbb{R} \in \mathfrak{X}$

ii) $B \in \mathfrak{X} \Rightarrow X^{-1}(B) \in \mathfrak{A} \Rightarrow (X^{-1}(B))^c = X^{-1}(B^c) \in \mathfrak{A} \Rightarrow B^c \in \mathfrak{X}$

iii) $\forall n \in \mathbb{N}: B_n \in \mathfrak{X} \Rightarrow \forall n \in \mathbb{N}: X^{-1}(B_n) \in \mathfrak{A} \Rightarrow \bigcup_{n\in\mathbb{N}} X^{-1}(B_n) = X^{-1}\left(\bigcup_{n\in\mathbb{N}} B_n\right) \in \mathfrak{A}$

$\Rightarrow \bigcup_{n\in\mathbb{N}} B_n \in \mathfrak{X}$

"$\Leftarrow$" $\forall \alpha \in \mathbb{R}: \{X \le \alpha\} = X^{-1}((-\infty,\alpha]) \in \mathfrak{A}$

$\Rightarrow \forall \alpha \ge \beta \in \mathbb{R}: \{X \le \alpha\} \cap \{X \le \beta\}^c = X^{-1}((\beta,\alpha]) \in \mathfrak{A}$

$\Rightarrow \forall \alpha \ge \beta \in \mathbb{R}: (\beta,\alpha] \in \mathfrak{X}$

$\Rightarrow \mathfrak{A}(\{(\beta,\alpha] \mid \alpha \ge \beta \in \mathbb{R}\}) = \mathcal{B}^1 \subseteq \mathfrak{X}$

$\Rightarrow \forall B \in \mathcal{B}^1: X^{-1}(B) \in \mathfrak{A}$

b) X sei ZV, $a,b \in \mathbb{R}$

z.z.: $a+bX$ ist ZV

<u>1. Fall:</u> $b=0$: zeige a ist ZV

Sei $B \in \mathcal{B}^1 \Rightarrow X^{-1}(B) = \left\{\begin{matrix} \emptyset & a \notin B \\ \Omega & a \in B \end{matrix}\right\} \in \mathfrak{A}$

<u>2. Fall:</u> $b \neq 0$:

X ZV $\overset{a)}{\Rightarrow} \forall \alpha \in \mathbb{R}: \{X \le \alpha\} \in \mathfrak{A}$

$\Rightarrow \forall \alpha \in \mathbb{R}: \{X \le b^{-1}(\alpha - a)\} \in \mathfrak{A}$

$\Rightarrow \forall \alpha \in \mathbb{R}: \{bX + a \le \alpha\} \in \mathfrak{A}$

$\overset{a)}{\Rightarrow} bX+a$ ist ZV

c) z.z.: $\forall \alpha \in \mathbb{R}: \{X^2 \le \alpha\} \in \mathfrak{A}$

<u>1. Fall:</u> $\alpha < 0$: $\{X^2 \le \alpha\} = \emptyset \in \mathfrak{A}$

<u>2. Fall:</u> $\alpha \ge 0$: $\{X^2 \le \alpha\} = \{X \le \sqrt{\alpha}\} \cap \{X \ge -\sqrt{\alpha}\} = \underbrace{X^{-1}(\underbrace{[-\sqrt{\alpha},\sqrt{\alpha}]}_{\in \mathcal{B}^1})}_{\in \mathfrak{A}}$

Aufgabe 15: $X \sim Bin(n,p)$

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\frac{P(X=k+1)}{P(X=k)} = \frac{\binom{n}{k+1} p^{k+1}(1-p)^{n-k-1}}{\binom{n}{k} p^k (1-p)^{n-k}} = \frac{p}{1-p} \frac{\frac{n!}{(k+1)!(n-k-1)!}}{\frac{n!}{k!(n-k)!}} = \frac{p}{1-p} \frac{n-k}{k+1} \overset{?}{=} 1$$

$$k' \in \mathbb{R}: \frac{(n-k')p}{(k'+1)(1-p)} = 1 \Rightarrow (k'+1)(1-p) = (n-k')p \Rightarrow k' = p(n+1) - 1$$

1. Fall: $k' \in \{0,1,2,\dots,n-1\}$

$\Rightarrow P(X=k) = P(X=k+1)$

$\Rightarrow$ doppeltes Maximum bei $k, k+1 \Rightarrow \exists 2$ Modalwerte

2. Fall: $k' \notin \mathbb{N}$, $k' \neq n$

$\Rightarrow$ Maximum bei $\lceil p(n+1)-1 \rceil = \lceil k' \rceil$

3. Fall: $k' = n \Rightarrow$ Max. bei n

Aufgabe 13:

b) $\Omega = \{ \omega = (\omega_1, \omega_2, \dots) \mid \omega_i \in \{0,1\} \}$

bel. Sequenz $m = (m_1, m_2, m_3, \dots, m_\ell)$, $\ell \in \mathbb{N}$

$A_n \hat{=}$ Sequenz m fällt ab dem n-ten Wurf

$= \{ \omega \in \Omega \mid \forall i \in \mathbb{N}, i \leq \ell : \omega_{n+i-1} = m_i \}$

Problem: $\{A_n\}_{n \in \mathbb{N}}$ sind nicht stochastisch unabhg.

$\{A_{n \cdot \ell}\}_{n \in \mathbb{N}}$ sind s.u.

$$P(A_n) = \left(\tfrac{1}{2}\right)^\ell ; \quad \sum_{n=1}^{\infty} P(A_n) = \infty \overset{\text{Borel}}{\underset{\text{Cantelli}}{\Rightarrow}} P(\limsup_{n \to \infty} A_n) = 1$$

$\Rightarrow$ bel. Muster tritt mit W'keit 1 ∞ oft auf.

a) $A_{(m)} \hat{=}$ Sequenz (m) tritt auf, $P(A_{(m)}) = 1$

$A \hat{=}$ alle Sequenzen treten auf

$A = \bigcap_{m \in M} A_{(m)}$ $\quad M$ ist abzählbar

$$P(A) = 1 - P(A^c) = 1 - P\Big(\big(\bigcap_{m \in M} A_{(m)}\big)^c\Big) = 1 - P\Big(\bigcup_{m \in M} A_{(m)}^c\Big) \geq 1 - \sum_{m \in M} \underbrace{P(A_{(m)}^c)}_{=0} = 1$$

Aufgabe 11:

Ereignisse: B_1: Urne 1 gewählt
B_2: " 2 "

A: m weiße und n-m schwarze Kugeln gezogen

gesucht: $P(B_i \mid A)$

gegeben: $P(A \mid B_i) = \dfrac{\binom{w_i}{m}\binom{s_i}{n-m}}{\binom{w_i+s_i}{n}}$

$$P(B_i \mid A) = \frac{P(A \mid B_i) \cdot P(B_i)}{P(A \mid B_1) \cdot P(B_1) + P(A \mid B_2) \cdot P(B_2)}$$

Bayes

Aufgabe 17:

a) $F_X: \mathbb{R} \to [0,1]$, $x \mapsto P(X \le x)$ $\qquad F_X(x) = P(X \le x) \in [0,1]$

i) $x < y: \quad F_X(x) = P(X \le x) = P^X((-\infty, x]) \overset{La 2.11c}{\le} P^X((-\infty, y]) = P(X \le y) = F_X(y)$

ii) $\lim\limits_{x\to\infty} F_X(x) = \lim\limits_{x\to\infty} P(X \le x) = \lim\limits_{x\to\infty} P^X((-\infty,x]) \overset{x_n = n}{=} \lim\limits_{n\to\infty} P^X((-\infty,n])$

$= \lim\limits_{n\to\infty} P^X\left(\bigcup_{i=1}^n (-\infty, i]\right) \overset{La 2.11}{=} P^X\left(\bigcup_{i\in\mathbb{N}} (-\infty, i]\right) = P^X(\mathbb{R}) = 1$

$\lim\limits_{x\to-\infty} F_X(x) = \lim\limits_{x\to-\infty} P^X((-\infty,x]) = \lim\limits_{n\to\infty} P^X((-\infty,-n]) = \lim\limits_{n\to\infty} P^X\left(\bigcap_{i=1}^n (-\infty,-i]\right)$

$\overset{La 2.11e}{=} P^X\left(\bigcap_{i\in\mathbb{N}} (-\infty,-i]\right) = P^X(\emptyset) = 0$

iii) $x_n \downarrow x_0$ für $n \to \infty$

$F_X(x_0) = P^X((-\infty, x_0]) = P^X\left(\bigcap_{n\in\mathbb{N}} (-\infty, x_n]\right) \overset{La 2.11e}{=} \lim\limits_{n\to\infty} P^X\left(\bigcap_{i=1}^n (-\infty, x_i]\right)$

$= \lim\limits_{n\to\infty} P^X((-\infty, x_n]) \underset{x_n \text{ bel.}}{=} \lim\limits_{x\downarrow x_0} P^X((-\infty,x]) = \lim\limits_{x\downarrow x_0} P(X \le x) = \lim\limits_{x\downarrow x_0} F_X(x)$

i)-iii) $\Rightarrow$ F_X ist Verteilungsfkt.!

b) $\forall x: \quad F_X(x) = P(X \le x) = P^X((-\infty,x]) = P^Y((-\infty,x]) = P(Y \le x) = F_Y(x)$

$\Rightarrow$ (da Def.bereich gleich) $F_X = F_Y$

Aufgabe 18:

$P(X > x+y \mid X > x) = P(X > y)$

$P^X((y,\infty)) = P^X((x+y,\infty) \mid (x,\infty)) = \dfrac{P^X((x+y,\infty)\cap(x,\infty))}{P^X((x,\infty))} = \dfrac{P^X((x+y,\infty))}{P^X((x,\infty))}$

$\Rightarrow P^X((x+y,\infty)) = P^X((x,\infty)) \cdot P^X((y,\infty))$

$P^X((x,\infty)) = 1 - P^X((-\infty,x]) = 1 - F_X(x) = \overline{F}_X(x)$

$\Rightarrow \overline{F}_X(x+y) = \overline{F}_X(x) \cdot \overline{F}_X(y) \qquad \forall x,y \ge 0 \qquad \overline{F}_X$ stetig

$\Rightarrow \exists \alpha \in \mathbb{R}: \quad \overline{F}_X(x) = e^{\alpha x}$

1. Fall: $\alpha > 0 \Rightarrow \overline{F}_X$ mon. steigend $\Rightarrow F_X$ mon. fallend $\lightning$ (Vert.fkt. $\nearrow$)

2. Fall: $\alpha = 0 \Rightarrow \overline{F}_X \equiv 1 \Rightarrow F_X \equiv 0$ $\lightning$ ($\lim\limits_{x\to\infty} F_X = 1$)

3. Fall: $\alpha < 0$, setze $\lambda = -\alpha \Rightarrow F_X = 1 - e^{-\lambda x}$ ist Vertfkt.

$\Rightarrow X \sim \text{Exp}(\lambda)$, $\lambda > 0 \Rightarrow$ alle(!) Gedächtnislosen Vert. sind Exp-Vert.

Aufgabe 19:

$X \sim Exp(\lambda)$, $\lambda = 0{,}01$, $f(x) = \lambda e^{-\lambda x} = 0{,}01 \cdot e^{-0{,}01x}$ $\forall x > 0$

$F(x) = P(X \leq x) = 1 - e^{-\lambda x} = 1 - e^{-0{,}01x}$ $\forall x > 0$

a) 1) $P(X > 100) = 1 - P(X \leq 100) = 1 - F(100) = 1 - 1 + e^{-1} = \frac{1}{e}$

2) $P(25 \leq X \leq 300) = P(X \leq 300) - P(X \underset{\text{da X abs. stetig}}{\overset{(<)}{\leq}} 25) = F(300) - F(25) = 1 - e^{-3} - 1 + e^{-\frac{1}{4}}$

$= e^{-\frac{1}{4}} - e^{-3}$

3) $P(X \leq 150) = F(150) = 1 - e^{-1{,}5} = 1 - \frac{1}{\sqrt{e^3}}$

b) $p = P(X \leq x) = F(x) = 1 - e^{-\lambda x}$

$\Leftrightarrow e^{-\lambda x} = 1 - p \Leftrightarrow -\lambda x = \ln(1-p) \Leftrightarrow x = \frac{\ln(1-p)}{-\lambda} = -100 \cdot \ln(1-p)$

$\Rightarrow x = -100 \cdot \ln(1 - 0{,}5) = -100 \cdot \ln \frac{1}{2} \approx 69{,}315$

c) $p = P(u_1 \leq X \leq u_2) = P(X \leq u_2) - P(X \leq u_1) = F(u_2) - F(u_1)$

$\Rightarrow p = 1 - e^{-\lambda u_2} - 1 + e^{-\lambda u_1} = e^{-\lambda u_1} - e^{-\lambda u_2}$

$\Rightarrow e^{-\lambda u_2} = e^{-\lambda u_1} - p \Leftrightarrow -\lambda u_2 = \ln(e^{-\lambda u_1} - p)$

$\Rightarrow u_2 = -\frac{1}{\lambda} \ln(e^{-\lambda u_1} - p)$

gesucht: minimale Intervallänge

$\min_{u_1 \geq 0} \left(-\frac{1}{\lambda} \ln(e^{-\lambda u_1} - p) - u_1\right) =: \min_{u_1} f(u_1)$

$f'(u_1) = \frac{1}{e^{-\lambda u_1} - p} e^{-\lambda u_1} - 1 \underset{x := e^{-\lambda u_1}}{=} \frac{x}{x-p} - 1 \overset{!}{\geq} 0$ (kein Extremum)

denn $x - p \leq 0 \Rightarrow e^{-\lambda u_1} \leq p$, aber $p = P(u_1 \leq X \leq u_2) < e^{-\lambda u_1}$ $\lightning$

also $\frac{x}{x-p} > 1 \Rightarrow f^*(u_1)$ monoton steigend $\Rightarrow$ wähle $u_1 = 0 \Rightarrow u_2 = -\frac{1}{\lambda} \ln(1-p$

Aufgabe 20:

$L \hat{=}$ Lebensdauer

Annahmen: $P(L<0)=0$

$P(L=0) = 0{,}05$

$P(L \leq x \mid L>0) = 1-e^{-\lambda x}, \quad x>0$

Verteilungsfunktion:

$x<0: \quad F(x)=0$

$x=0: \quad F(x)=0{,}05$

$x>0: \quad F(x) = P(L\leq x) = P(L\leq x \mid L\leq 0)\cdot P(L\leq 0) + P(L\leq x \mid L>0)\cdot P(L>0)$

$= 1\cdot 0{,}05 + (1-e^{-\lambda x})\cdot 0{,}95$

$\Rightarrow F(x) = 0{,}05 + (1-e^{-\lambda x})0{,}95 \cdot \mathbb{1}_{[0,\infty)}(x)$

weder diskret noch abs.-stetig

$\uparrow$ $\nexists$ abzählbarer Träger

$\uparrow$ $\lim\limits_{x\downarrow 0} F(x) = 0 \neq 0{,}05$

Aufgabe 21:

$X \hat{=}$ Augenzahl von Würfel 1

$Y \hat{=}$ —— " —— 2

$S = X+Y$

Bedingung: $P(S=s) = \frac{1}{11}, \quad s\in\{2,3,\dots,12\}$

$P(X=i) = x_i, \quad P(Y=i) = y_i \qquad i\in\{1,\dots,6\}$

$$\left.\begin{array}{l} P(S=2) = x_1y_1 \\ P(S=12) = x_6y_6 \end{array}\right\} x_1y_1 = x_6y_6 = \frac{1}{11} \Rightarrow \frac{y_6}{y_1} = \frac{x_1}{x_6} =: z > 0$$

$$\frac{1}{11} = P(S=7) = x_1y_6 + x_2y_5 + x_3y_4 + x_4y_3 + x_5y_2 + x_6y_1$$

$$\geq x_1y_6 + x_6y_1 = \frac{x_1}{x_6}y_6x_6 + \frac{y_1}{y_6}y_6x_6 = \Big(\underbrace{\frac{x_1}{x_6}}_{z} + \frac{y_1}{y_6}\Big)y_6x_6$$

$$= \frac{1}{11}\left(z+\frac{1}{z}\right)$$

also $\frac{1}{11} \geq \frac{1}{11}\left(z+\frac{1}{z}\right) \Rightarrow 1 \geq \left(z+\frac{1}{z}\right)$ $\lightning\,(z>0)$

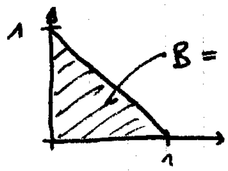
Aufgabe 22:

$X_1 \sim Exp(\lambda_1)$ $\qquad$ X_1, X_2 sind s.u.

$X_2 \sim Exp(\lambda_2)$

a) $f_{(X_1,X_2)}(x_1,x_2) = f_{X_1}(x_1) \cdot f_{X_2}(x_2) = (\lambda_1 e^{-\lambda_1 x_1})(\lambda_2 e^{-\lambda_2 x_2})$

b) $P(X_1 + X_2 \leq 1)$



$B = \{(x_1,x_2) \mid x_1, x_2 \geq 0 \wedge x_1 + x_2 \leq 1\}$

$$p = P((x_1 x_2) \in B) = \int_0^1 \int_0^{1-x_2} f_{(X_1,X_2)}(x_1,x_2)\, dx_1\, dx_2$$

$$= \int_0^1 \int_0^{1-x_2} (\lambda_1 e^{-\lambda_1 x_1})(\lambda_2 e^{-\lambda_2 x_2})\, dx_1\, dx_2$$

$$= \lambda_1 \lambda_2 \int_0^1 e^{-\lambda_2 x_2} \left(\int_0^{1-x_2} e^{-\lambda_1 x_1}\, dx_1 \right) dx_2$$

$$= \lambda_1 \lambda_2 \int_0^1 e^{-\lambda_2 x_2} \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_1} e^{-\lambda_1 (1-x_2)} \right) dx_2$$

$$= \lambda_2 \int_0^1 e^{-\lambda_2 x_2}\, dx_2 - \lambda_2 \int_0^1 e^{-\lambda_2 x_2 - \lambda_1 + \lambda_1 x_2}\, dx_2$$

$$= 1 - e^{-\lambda_2} - \lambda_2 \int_0^1 e^{-\lambda_1}\, e^{x_2(\lambda_1 - \lambda_2)}\, dx_2$$

$\lambda_1 \neq \lambda_2$:

$$= 1 - e^{-\lambda_2} - \frac{\lambda_2}{\lambda_1 - \lambda_2}(e^{-\lambda_2} - e^{-\lambda_1})$$

$$= 1 - \frac{\lambda_1 - \lambda_2}{\lambda_1 - \lambda_2} e^{-\lambda_2} - \frac{\lambda_2}{\lambda_1 - \lambda_2}(e^{-\lambda_2} - e^{-\lambda_1})$$

$$= 1 - \frac{1}{\lambda_1 - \lambda_2}(\lambda_1 e^{-\lambda_2} - \lambda_2 e^{-\lambda_1})$$

$\lambda_1 = \lambda_2 = \lambda$:

$$= 1 - e^{-\lambda} - \lambda \int e^{-\lambda}\, dx_2$$

$$= 1 - e^{-\lambda} - \lambda e^{-\lambda}$$

Aufgabe 23:

"$\Rightarrow$" $X_1, \dots, X_n$ s.u.

Beh. $P(X_1 = t_1, \dots, X_{j-1} = t_{j-1}, X_j \le t_j, \dots, X_n \le t_n)$

$$= \prod_{i=1}^{j-1} P(X_i = t_i) \cdot \prod_{i=j}^{n} P(X_i \le t_i)$$

z.z. gilt für $j = n$

Beweis: Induktion (o.BdA. Träger $\mathbb{N}$)

IA: $j=1$ gilt nach Voraussetzung

IS: $j \to j+1$

$$P(X_1 = t_1, \dots, X_j = t_j, X_{j+1} \le t_{j+1}, \dots, X_n \le t_n)$$

$$= P(X_1 = t_1, \dots, X_{j-1} = t_{j-1}, X_j \le t_j, \dots, X_n \le t_n)$$

$$- P(X_1 = t_1, \dots, X_{j-1} = t_{j-1}, \underset{X_j < t_j}{X_j \le t_j - 1}, X_{j+1} \le t_{j+1}, \dots, X_n \le t_n)$$

$$\overset{(IV)}{=} \prod_{i=1}^{j-1} P(X_i = t_i) \cdot P(X_j \le t_j) \prod_{i=j+1}^{n} P(X_i \le t_i)$$

$$- \prod_{i=1}^{j-1} P(X_i = t_i) \cdot P(X_j \le t_j - 1) \cdot \prod_{i=j+1}^{n} P(X_i \le t_i)$$

$$= \prod_{i=1}^{j-1} P(X_i = t_i) \prod_{i=j+1}^{n} P(X_i \le t_i) \cdot \underbrace{\left(P(X_j \le t_j) - P(X_j \le t_j - 1)\right)}_{P(X_j = t_j)}$$

$$= \prod_{i=1}^{j} P(X_i = t_i) \cdot \prod_{i=j+1}^{n} P(X_i \le t_i)$$

$\Rightarrow$ Beh.

"$\Leftarrow$" $P(X_1 \le t_1, \dots, X_n \le t_n) = \sum_{\substack{s_i \in T_i : s_i \le t_i \\ i = \{1, \dots, n\}}} P(X_1 = s_1, \dots, X_n = s_n)$

$$\overset{\text{Vor.}}{=} \sum_{s_i \le t_i} \prod_{i=1}^{n} P(X_i = s_i)$$

$$= \prod_{i=1}^{n} \sum_{s_i \le t_i} P(X_i = s_i) = \prod_{i=1}^{n} P(X_i \le t_i) \Rightarrow \text{s.u.} \quad \square$$

Aufgabe 24:

$N \sim Poi(\lambda)\ ,\ \lambda = 5$

$T \sim Exp(\mu)\ ,\ \mu = 0{,}01$

a) $P(N<4,\ T<50) = P(N<4)\cdot P(T<50) = P(N\in\{0,1,2,3\})\cdot P(T\in[0,50))$

$$= \left(e^{-\lambda}\sum_{i=0}^{3}\frac{\lambda^i}{i!}\right)\left(1-e^{-50\mu}\right) \approx 0{,}10428$$

b) $P(N\leq 12,\ NT\leq 300) = \sum_{k=0}^{12} P(N=k,\ NT\leq 300) = \sum_{k=0}^{12} P(N=k,\ kT\leq 300)$

$$= P(N=0) + \sum_{k=1}^{12} P(N=k)\cdot P\left(T\leq \frac{300}{k}\right)$$

$$= e^{-\lambda} + e^{-\lambda}\sum_{k=1}^{12}\frac{\lambda^k}{k!}\left(1-e^{-\frac{300}{k}\mu}\right) \approx 0{,}4967$$

Aufgabe 25:

$\Gamma(\alpha,\lambda) * \Gamma(\beta,\lambda)\ ;\quad X_1\sim\Gamma(\alpha,\lambda),\ X_2\sim\Gamma(\beta,\lambda)$

$y<0:\quad P(X_1+X_2<y) = 0 \quad\Rightarrow\quad f_{X_1+X_2}(y) = 0$

$y\geq 0:\quad f_{X_1+X_2}(y) = \int_{-\infty}^{\infty} f_{X_1}(t)\, f_{X_2}(y-t)\, dt = \int_{-\infty}^{\infty}\frac{\lambda^\alpha}{\Gamma(\alpha)} t^{\alpha-1} e^{-\lambda t}\, 1_{(0,\infty)}(t)\cdot\frac{\lambda^\beta}{\Gamma(\beta)}(y-t)^{\beta-1} e^{-\lambda(y-t)}\, 1_{(0,\infty)}(y-t)$

$$= \int_0^y \frac{\lambda^{(\alpha+\beta)}}{\Gamma(\alpha)\Gamma(\beta)}\, e^{-\lambda t-\lambda(y-t)}\, t^{\alpha-1}(y-t)^{\beta-1}\, dt$$

$$= \frac{\lambda^{(\alpha+\beta)}}{\Gamma(\alpha)\Gamma(\beta)}\cdot e^{-\lambda y}\int_0^y t^{\alpha-1}(y-t)^{\beta-1}\, dt$$

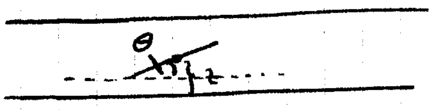
$$\overset{t=s\cdot y}{=} \frac{\lambda^{(\alpha+\beta)}}{\Gamma(\alpha)\Gamma(\beta)}\, e^{-\lambda y}\int_0^1 (sy)^{\alpha-1}(y-sy)^{\beta-1}\, y\, ds$$

$$= \frac{\lambda^{(\alpha+\beta)}}{\Gamma(\alpha)\cdot\Gamma(\beta)}\cdot e^{-\lambda y}\, y^{\alpha-1}\, y^{\beta-1}\, y\, \underbrace{\int_0^1 s^{\alpha-1}(1-s)^{\beta-1}\, ds}_{=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}}$$

$$= \frac{\lambda^{(\alpha+\beta)}}{\Gamma(\alpha+\beta)}\cdot e^{-\lambda y}\cdot y^{(\alpha+\beta)-1}$$

$$\Rightarrow\quad X_1+X_2 \overset{*}{\sim} \Gamma(\alpha+\beta,\lambda) = \Gamma(\alpha,\lambda) * \Gamma(\beta,\lambda)$$

Aufgabe 26:



$z \sim R(0,1)$

$\theta \sim R(0,\pi)$

$$f_{(Z,\Theta)}(z,\vartheta) = 1 \cdot 1\!\!1_{(0,1)}(z) \cdot \frac{1}{\pi} 1\!\!1_{(0,\pi)}(\vartheta) = \frac{1}{\pi} \cdot 1\!\!1_{((0,1)\times(0,\pi))}(z,\vartheta)$$

$B \subseteq [0,1] \times [0,\pi]$ (Ränder haben W'keit 0, also offen/geschl. egal)

$\hat{=}$ Nadel schneidet Linie

$$B = \{ (z,\vartheta) \mid z \le \tfrac{1}{2}\sin\vartheta \vee 1-z \le \tfrac{1}{2}\sin\vartheta \}$$

$$P(B) = \iint_B f_{(Z,\Theta)}(z,\vartheta)\, dz\, d\vartheta = \frac{1}{\pi} \int_0^\pi \left(\int_0^{\frac{1}{2}\sin\vartheta} dz + \int_{1-\frac{1}{2}\sin\vartheta}^{1} dz \right) d\vartheta = \frac{1}{\pi} \int_0^\pi \left(\tfrac{1}{2}\sin\vartheta + 1 - 1 + \tfrac{1}{2}\sin\vartheta \right) d\vartheta$$

$$= \frac{1}{\pi} \int_0^\pi \sin\vartheta \, d\vartheta = \frac{1}{\pi} \left[-\cos\vartheta \right]_0^\pi = \frac{2}{\pi}$$

Aufgabe 27:

a) $X \sim R(0,1)$

$Y = e^{-X^2}$ $\quad T(x) = e^{-x^2}$

$T((0,1)) = (\frac{1}{e}, 1) \neq (-\infty,\infty)$ nicht Träger der Normalverteilung $\lightning$

$T^{-1}(y) = \sqrt{-\ln y}\,, \quad y \in (\frac{1}{e},1)$

$\det\left[\frac{\partial T(x)}{\partial x}\right] = -2xe^{-x^2} \neq 0 \quad \forall x>0$ stetig, ...

Transformationssatz: $$f_Y(y) = \frac{f_X(T^{-1}(y))}{\left|\det\left(\frac{\partial T(x)}{\partial x}\right)\Big|_{T^{-1}(y)}\right|} 1\!\!1_{T((0,1))}(y) = \frac{f_X(\sqrt{-\ln y})}{\left|-2xe^{-x^2}\big|_{\sqrt{-\ln y}}\right|} \cdot 1\!\!1_{(\frac{1}{e},1)}(y)$$

$$= \frac{1}{2\sqrt{-\ln y}\cdot e^{-\sqrt{-\ln y}^2}} 1\!\!1_{(\frac{1}{e},1)}(y) = \frac{1}{2y\sqrt{-\ln y}} 1\!\!1_{(\frac{1}{e},1)}(y)$$

b) $X_1, X_2 \sim R(0,1)$

$Y_1 = \sqrt{-2\ln X_1}\,\sin(2\pi X_2)\,, \quad Y_2 = \sqrt{-2\ln X_1}\,\cos(2\pi X_2)$

$Q = (0,1)^2\,, \quad X = (X_1, X_2) \quad f_X(x_1,x_2) = 1\!\!1_Q(x_1,x_2)$

$T(x_1,x_2) = \left(\sqrt{-2\ln x_1}\,\sin(2\pi x_2)\,, \sqrt{-2\ln x_1}\,\cos(2\pi x_2)\right) \quad (x_1,x_2) \in Q$

$$\det\left(\frac{\partial T_i}{\partial x_j}\right) = \det\begin{pmatrix} \frac{-\sin(2\pi x_2)}{\sqrt{-2\ln x_1}\; x_1} & 2\pi\sqrt{-2\ln x_1}\cdot\cos(2\pi x_2) \\ \frac{-\cos(2\pi x_2)}{\sqrt{-2\ln x_1}\; x_1} & -2\pi\sqrt{-2\ln x_1}\;\sin(2\pi x_2) \end{pmatrix}$$

$$= \frac{2\pi}{x_1}\sin^2(2\pi x_2) + \frac{2\pi}{x_1}\cos^2(2\pi x_2) = \frac{2\pi}{x_1} \neq 0 \quad \text{für } \forall x \in (0,1)$$

Transform.satz:

T^{-1}: $Y_1^2 = -2\ln x_1 \sin^2(2\pi x_2)$ $\qquad$ $Y_2^2 = -2\ln x_1 \cos^2(2\pi x_2)$

$\Rightarrow Y_1^2 + Y_2^2 = -2\ln x_1$

$\Rightarrow x_1 = e^{-\frac{1}{2}(y_1^2+y_2^2)}$

$x_2 = \ldots$ (wird für Rechnung nicht benötigt)

$$f_Y(y) = \frac{f_X(T^{-1}(y))}{\left|\det(\ldots)\big|_{T^{-1}(y)}\right|}\,\mathbb{1}_{T(Q)}(y) = \frac{e^{-\frac{1}{2}(y_1^2+y_2^2)}}{2\pi}\,\mathbb{1}_{\mathbb{R}^2}(y_1,y_2)$$

Aufgabe 28:

a) $\operatorname{Bin}(n_1,p) * \operatorname{Bin}(n_2,p) \qquad n_1,n_2 \in \mathbb{N}, \quad 0 \le p \le 1$

Zeige zunächst:

$$\operatorname{Bin}(n,p) * \operatorname{Bin}(1,p) \stackrel{!}{=} \operatorname{Bin}(n+1,p)$$

$$f_{X_1+X_2}(k) = \sum_{i=0}^{k} f_{X_1}(i) \underbrace{f_{X_2}(k-i)}_{\neq 0 \text{ für } i \in \{k,k-1\}}$$

$$= \binom{n}{k} p^k (1-p)^{n-k}(1-p) + \binom{n}{k-1} p^{k-1}(1-p)^{n-k+1} p$$

$$= p^k (1-p)^{n-k+1}\left[\binom{n}{k}+\binom{n}{k-1}\right]$$

$$= \binom{n+1}{k} p^k (1-p)^{n+1-k} \quad \Rightarrow \quad \operatorname{Bin}(n+1,p)\text{ - verteilt}$$

$$\Rightarrow \operatorname{Bin}(n_1,p) = \underbrace{\operatorname{Bin}(1,p) * \ldots * \operatorname{Bin}(1,p)}_{n_1\text{-mal}}$$

$$\operatorname{Bin}(n_1,p) * \operatorname{Bin}(n_2,p) = \underbrace{\operatorname{Bin}(1,p) * \ldots * \operatorname{Bin}(1,p)}_{n_1\text{-mal}} * \underbrace{\operatorname{Bin}(1,p) * \ldots * \operatorname{Bin}(1,p)}_{n_2\text{-mal}}$$

$$= \operatorname{Bin}(n_1+n_2,p)$$

b) $\operatorname{Poi}(\lambda_1) * \operatorname{Poi}(\lambda_2) \qquad \lambda_1,\lambda_2 > 0 \qquad$ also $\quad X_1 \sim \operatorname{Poi}(\lambda_1),\ X_2 \sim \operatorname{Poi}(\lambda_2)$

$$f_{X_1+X_2}(k) = \sum_{i=0}^{k} f_{X_1}(i)\, f_{X_2}(k-i)$$

$$= \sum_{i=0}^{k} e^{-\lambda_1}\frac{\lambda_1^i}{i!} e^{-\lambda_2}\frac{\lambda_2^{(k-i)}}{(k-i)!} = e^{-(\lambda_1+\lambda_2)} \sum_{i=0}^{k} \frac{\lambda_1^i \lambda_2^{k-i}}{i!\cdot(k-i)!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{k!} \sum_{i=0}^{k} \binom{k}{i} \lambda_1^i \lambda_2^{k-i} = \frac{e^{-(\lambda_1+\lambda_2)}}{k!}\cdot(\lambda_1+\lambda_2)^k$$

$$\Rightarrow X_1+X_2 \sim \operatorname{Poi}(\lambda_1+\lambda_2)$$

c) $X_1,\ldots,X_n \sim \operatorname{Geo}(p)$ s.u. $\quad 0 < p < 1$

z.z.: $\quad P\left(\sum_{i=1}^{n} X_i = k\right) = \binom{n+k-1}{n-1}(1-p)^k p^n$

Also: $\quad \underbrace{\operatorname{Geo}(p) * \ldots * \operatorname{Geo}(p)}_{n} = \overline{\operatorname{Bin}}(n,p)$

Bew. (Induktion):

$n=1: \quad P(X_1 = k) = \binom{k}{0}(1-p)^k p$

$n \to n+1: \quad P\left(\sum_{i=1}^{n+1} X_i = k\right) = \sum_{j=0}^{k} P\left(\sum_{i=1}^{n} X_i = j\right)\cdot P(X_{n+1} = k-j)$

$$\overset{I.V.}{=} \sum_{j=0}^{k} \binom{n+j-1}{n-1} (1-p)^j p^n (1-p)^{k-j} p = (1-p)^k p^{n+1} \sum_{j=0}^{k} \binom{n+j-1}{n-1}$$

$$\underset{\left(\substack{\text{Ind. über} \\ k}\right)}{=} \binom{n+k}{n} (1-p)^k p^{n+1} \quad \checkmark$$

d) $\overline{Bin}(n_1,p) * \overline{Bin}(n_2,p) \qquad n_1,n_2 \in \mathbb{N}, \quad 0<p<1$

$$\overline{Bin}(n_1,p) * \overline{Bin}(n_2,p) \overset{c)}{=} \underbrace{Geo(p) * \ldots * Geo(p)}_{n_1\text{-mal}} * \underbrace{Geo(p) * \ldots * Geo(p)}_{n_2\text{-mal}}$$

$$\overset{c)}{=} \overline{Bin}(n_1+n_2,p)$$

Aufgabe 29:

$N(s)$: Anzahl der Programme, die zur Zeit $t+s$ abgearbeitet sind

$N(s) = \max\{ n \in \mathbb{N} \mid \sum_{i=1}^{n} X_i \le s \}$ mit $X_i \sim Exp(2)$ s.u.

$\Rightarrow$ Poisson-Prozess

$\overset{La.\,5.7.}{\Rightarrow} N(s) \sim Poi(2s)$ mit $s=6$

$\Rightarrow N(6) \sim Poi(12)$

$$0{,}95 \le P(N(6) \ge n) = e^{-12} \sum_{i=n}^{\infty} \frac{12^i}{i!} = 1 - e^{-12} \sum_{i=0}^{n-1} \frac{12^i}{i!}$$

$$\Leftrightarrow e^{-12} \sum_{i=0}^{n-1} \frac{12^i}{i!} \le 0{,}05$$

$n=7$: $e^{-12} \sum_{i=0}^{6} \frac{12^i}{i!} = 0{,}046 < 0{,}05$

$n=8$: $e^{-12} \sum_{i=0}^{7} \frac{12^i}{i!} = 0{,}0895 > 0{,}05$

$\Rightarrow n \le 7$

Alternativ:

T_n : Bearbeitungszeit für n Programme

$T_n = \sum_{i=1}^{n} X_i \sim \Gamma(n,2)$

$$0{,}95 \le P(T_n \le 6) = \int_0^6 \frac{2^n}{(n-1)!} x^{n-1} e^{-2x}\,dx \overset{!}{=} 1 - \sum_{i=0}^{n-1} \frac{12^i}{i!} e^{-12}$$

Induktion

Aufgabe 30:

a) $Y = X_1 X_2$

Setze: $T: \begin{cases} (0,\infty)^2 \longrightarrow (0,\infty)^2 \\ (x_1,x_2) \longmapsto (x_1 \cdot x_2, x_2) \end{cases}$

$\Rightarrow \quad T^{-1}(y_1,y_2) = \left(\frac{y_1}{y_2}, y_2\right)$

$T^{-1}(T(x_1,x_2)) = T^{-1}(x_1 x_2, x_2) = \left(\frac{x_1 x_2}{x_2}, x_2\right) = (x_1, x_2)$ da $x_2 > 0$

$T(T^{-1}(y_1,y_2)) = T\left(\frac{y_1}{y_2}, y_2\right) = (y_1, y_2)$

$\Rightarrow$ T bijektiv auf $(0,\infty)^2$

$$\det\left(\frac{\partial T_i}{\partial x_j}\right) = \begin{vmatrix} x_2 & x_1 \\ 0 & 1 \end{vmatrix} = x_2 \neq 0$$

$\Rightarrow$ (Trafo-Satz) $(X_1 X_2, X_2)$ ist abs. stetig mit Dichte:

$$f_{(X_1X_2,X_2)}(y,t) = \frac{1}{t} f_{(X_1,X_2)}\left(\frac{y}{t}, y\right)$$

$$f_{(X_1X_2)}(y) = \int_0^\infty \frac{1}{t} f_{(X_1,X_2)}\left(\frac{y}{t}, t\right) dt \underset{\text{s.u.}}{=} \int_0^\infty \frac{1}{t} f_{X_1}\left(\frac{y}{t}\right) \cdot f_{X_2}(t)\, dt$$

b) $Z = \frac{X_1}{X_2}$ $\quad T: \begin{cases} (0,\infty)^2 \longrightarrow (0,\infty)^2 \\ (x_1,x_2) \longrightarrow \left(\frac{x_1}{x_2}, x_2\right) \end{cases}$

ist bijektiv (s. a))

$$\det\left(\frac{\partial T_i}{\partial x_j}\right) = \begin{vmatrix} \frac{1}{x_2} & -\frac{x_1}{x_2^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{x_2} \neq 0$$

$$\Rightarrow \quad f_{\left(\frac{X_1}{X_2}, X_2\right)}(z,t) = t \cdot f_{(X_1,X_2)}(z \cdot t, t)$$

$$\Rightarrow \quad f_{\frac{X_1}{X_2}}(z) = \int_0^\infty t\, f_{X_1}(z \cdot t) \cdot f_{X_2}(t)\, dt$$

Aufgabe 32:

a) z. z. $E[aX+bY] = aE[X] + bE[Y]$

X, Y abs. stetig:

$$E[aX+bY] = \iint (ax+by)\, f_{(X,Y)}(x,y)\,dx\,dy$$
$$= a\iint x f_{(X,Y)}(x,y)\,dx\,dy + b\iint y\, f_{(X,Y)}(x,y)\,dx\,dy$$
$$= a\int x f_X(x)\,dx + b\int y\, f_Y(y)\,dy$$
$$= aE[X] + bE[Y]$$

X, Y diskret:

$$E[aX+bY] = \sum_x \sum_y (ax+by)\, P(X=x, Y=y)$$
$$= a\sum_x \sum_y x\, P(X=x, Y=y) + b\sum_x \sum_y y\, P(X=x, Y=y)$$
$$= a\sum_x x\, P(X=x) + b\sum_y y\cdot P(Y=y)$$
$$= aE[X] + bE[Y]$$

b) $X \leq Y \Rightarrow Y - X \geq 0$

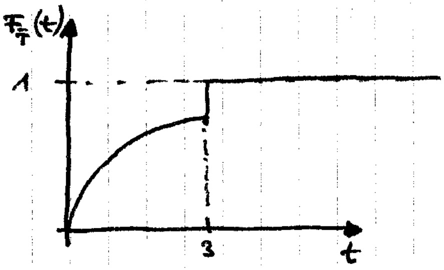
$$E[Y-X] = \begin{cases} \sum_x \sum_y (y-x)\, P(X=x, Y=y) \geq 0 \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (y-x)\, f_{(X,Y)}(x,y)\,dx\,dy \geq 0 \end{cases}$$

$$\overset{a)}{\Rightarrow} E[Y-X] = E[Y] - E[X] \geq 0 \Rightarrow E[Y] \geq E[X] \quad \square$$

Aufgabe 33:

$T \sim \mathrm{Exp}(\frac{1}{2})$

$\Rightarrow E[T] = 2$



nach La. 6.2:

$$E[X] = -\int_{-\infty}^{0} F(x)\,dx + \int_{0}^{\infty} (1 - F(x))\,dx$$

$$F_{\bar{T}}(x) = \begin{cases} 0 & x \leq 0 \\ 1-e^{-\lambda x} & 0 \leq x < 3 \\ 1 & x \geq 3 \end{cases}$$

$$\Rightarrow E[\bar{T}] = \int_0^3 (1 - 1 + e^{-\lambda x})\,dx + \int_3^{\infty} 0\,dx = \int_0^3 e^{-\lambda x}\,dx = 2(1 - e^{-1{,}5}) \approx 1{,}5537$$

$$\frac{E[\bar{T}]}{E[T]} = 0{,}7769$$

Aufgabe 34:

$X, Y \sim Exp(\lambda)$ i.i.d.

z.z.: $E[\min\{X,Y\}] \neq \min\{E[X], E[Y]\}$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \overset{\shortparallel}{E[X] = E[Y] = \frac{1}{\lambda}}$ (da X,Y gleichverteilt)

1. Mögl.:

$$E[\min\{X,Y\}] = \int_0^\infty\int_0^\infty \min\{x,y\}\,\lambda e^{-\lambda x}\,\lambda e^{-\lambda y}\,dx\,dy = \int_0^\infty\int_0^Y x\lambda^2 e^{-\lambda x}e^{-\lambda y}\,dx\,dy + \int_0^\infty\int_Y^\infty y\lambda^2 e^{-\lambda x}e^{\lambda y}\,dx\,dy$$

$$= \ldots$$

2. Mögl.:

$$P(\min\{X,Y\} \leq x) = 1 - P(X > x, Y > x) \overset{\text{s.u.}}{=} 1 - P(X>x)\cdot P(Y>x) \overset{\text{gleich-verteilt}}{=} 1 - (P(X>x))^2$$

$$= 1 - e^{-\lambda x}e^{-\lambda x} = 1 - e^{-2\lambda x}$$

$\Rightarrow \min\{X,Y\} \sim Exp(2\lambda)$

$\Rightarrow E[\min\{X,Y\}] = \frac{1}{2\lambda} \neq \frac{1}{\lambda}$

Aufgabe 31:

d) $$E[X] = \int_{-\infty}^\infty x\cdot\frac{1}{\pi(1+x^2)}\,dx = \lim_{t\to\infty}\int_{-t}^t \frac{x}{\pi(1+x^2)}\,dx \overset{\tau = 1+x^2}{=} \lim_{t\to\infty}\left[\ln\frac{1}{2}(1+x^2)\right]_{-t}^t = 0$$

$$E[|X|] = 2\cdot\int_0^\infty \frac{x}{\pi(1+x^2)}\,dx = \infty \implies \nexists\ E\text{-Wert},\ \nexists\ \text{Varianz}$$

c) $X \sim \Gamma(\alpha, \lambda)$

Berechne Laplace-Transf.

$$L(s) = \int_0^\infty e^{-sx}f(x)\,dx = \int_0^\infty e^{-sx}\frac{\lambda^\alpha}{\Gamma(\alpha)}\cdot x^{\alpha-1}e^{-\lambda x}\,dx = \frac{\lambda^\alpha}{\Gamma(\alpha)}\int_0^\infty x^{\alpha-1}e^{-(\lambda+s)x}\,dx$$

subst. $(\lambda+s)x =: y$

$$\frac{\lambda^\alpha}{\Gamma(\alpha)}\int_0^\infty\left(\frac{y}{\lambda+s}\right)^{\alpha-1}e^{-y}\frac{1}{\lambda+s}\,dy = \frac{\lambda^\alpha}{\Gamma(\alpha)(\lambda+s)^\alpha}\underbrace{\int_0^\infty y^{\alpha-1}e^{-y}\,dy}_{\Gamma(\alpha)} = \left(\frac{\lambda}{\lambda+s}\right)^\alpha$$

$$E[X] = -L'(0) = \left[-\alpha\left(\frac{\lambda}{\lambda+s}\right)^{\alpha-1}\left(-\frac{\lambda}{(\lambda+s)^2}\right)\right]_{s=0} = \frac{\alpha}{\lambda}$$

$$E[X^2] = +L''(0) = \left(-\alpha\frac{\lambda^\alpha}{(\lambda+s)^{\alpha+1}}\right)'_{s=0} = \left(-\alpha\lambda^\alpha(-\alpha-1)\frac{1}{(\lambda+s)^{\alpha+2}}\right)_{s=0} = \frac{\alpha(\alpha+1)}{\lambda^2}$$

$$\Rightarrow Var(X) = E[X^2] - (E[X])^2 = \frac{\alpha}{\lambda^2}$$

Aufgabe 31:

a) $X \sim Poi(\lambda)$

$$E(X) = \sum_{i=0}^{\infty} i\, P(X=i) = \sum_{i=0}^{\infty} i \cdot \frac{\lambda^i}{i!} e^{-\lambda} = e^{-\lambda} \sum_{i=1}^{\infty} \frac{\lambda^i}{(i-1)!} = \lambda e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = \lambda e^{-\lambda} \cdot e^{\lambda} = \lambda$$

$$Var(X) = E(X^2) - (E(X))^2$$

$$E(X^2) = \sum_{i=0}^{\infty} i^2\, P(X=i) = \sum_{i=0}^{\infty} i^2 \cdot \frac{\lambda^i}{i!} e^{-\lambda} = \lambda e^{-\lambda} \sum_{i=1}^{\infty} \frac{i \cdot \lambda^{i-1}}{(i-1)!} = \lambda e^{-\lambda} \sum_{i=1}^{\infty} \frac{\frac{d}{di} \lambda^i}{(i-1)!}$$

$$= \lambda e^{-\lambda} \frac{d}{d\lambda} \sum_{i=1}^{\infty} \frac{\lambda^i}{(i-1)!} = \lambda e^{-\lambda} \frac{d}{d\lambda} \lambda \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = \lambda e^{-\lambda} \frac{d}{d\lambda} \lambda e^{\lambda}$$

$$= \lambda e^{-\lambda} \left(e^{\lambda} + \lambda e^{\lambda} \right) = \lambda + \lambda^2$$

$$\Rightarrow Var(X) = \lambda + \lambda^2 - \lambda^2 = \lambda$$

b) $X \sim \overline{Bin}(n,p) = \underbrace{Geo(p) * \dots * Geo(p)}_{n}$

$Y_i \sim Geo(p) \qquad i = 1, \dots, n \qquad$ i.i.d.

$X = \sum_{i=1}^{n} Y_i \quad , \quad q := 1-p$

$$E(Y_1) = \sum_{k=0}^{\infty} k\, P(Y_1 = k) = \sum_{k=0}^{\infty} k\, (1-p)^k p = p \cdot q \sum_{k=1}^{\infty} k\, q^{k-1} = p \cdot q \frac{d}{dq} \sum_{k=1}^{\infty} q^k$$

$$= p \cdot q \frac{d}{dq} \sum_{k=0}^{\infty} q^{k+1} = p \cdot q \cdot \frac{d}{dq} \cdot \frac{q}{1-q} = pq \underbrace{\frac{1-q+q}{(1-q)^2}}_{p^2} = \frac{q}{1-q} = \frac{1-p}{p}$$

$$E(X) = E\left(\sum_{i=1}^{n} Y_i\right) = \sum_{i=1}^{n} E(Y_i) = n \cdot E(Y_i) = n \cdot \frac{1-p}{p}$$

$$Var(Y_i) = E(Y_i^2) - (E(Y_i))^2$$

$$E(Y_1^2) = E(Y_1(Y_1 - 1) + Y_1) = E(Y_1(Y_1-1)) + E(Y_1)$$

$$E(Y_1(Y_1-1)) = \sum_{k=0}^{\infty} k(k-1)\, P(Y_1 = k) = \sum_{k=2}^{\infty} k(k-1)\, q^k p = pq^2 \cdot \sum_{k=2}^{\infty} k(k-1)\, q^{k-2}$$

$$= pq^2 \sum_{k=2}^{\infty} \frac{d^2}{dq^2} q^k = pq^2 \frac{d^2}{dq^2} \sum_{k=0}^{\infty} q^{k+2} = pq^2 \frac{d^2}{dq^2} q^2 \sum_{k=0}^{\infty} q^k$$

$$= pq^2 \frac{d^2}{dq^2} \frac{q^2}{1-q} = pq^2 \frac{d}{dq} \frac{2q(1-q) + q^2}{(1-q)^2}$$

$$= pq^2 \frac{d}{dq} \frac{2q - q^2}{(1-q)^2} = pq^2 \cdot \frac{(2-2q)(1-q)^{\cancel{2}} - 2\cancel{(1-q)}(-1)(2q-q^2)}{(1-q)^{\cancel{4}\,3}}$$

$$= pq^2 \cdot \frac{2 - 2q - 2q + 2q^2 + 4q - 2q^2}{(1-q)^3} = \frac{2q^2}{(1-q)^2} = \frac{2(1-p)^2}{p^2}$$

$$\Rightarrow E(Y_1^2) = \frac{2(1-p)^2}{p^2} + \frac{1-p}{p} =$$

$$\Rightarrow \mathrm{Var}(Y_i) = E(Y_i^2) - (E(Y_i))^2 = \frac{2(1-p)^2}{p} + \frac{1-p}{p} - \left(\frac{1-p}{p}\right)^2$$

$$= \frac{2(1-p)^2 + p(1-p) - (1-p)^2}{p^2} = \frac{(1-p)^2 + p(1-p)}{p^2} = \frac{1-p}{p^2}$$

$$\mathrm{Var}(X) = \mathrm{Var}\left(\sum_{i=1}^{n} Y_i\right) \overset{Y_i \text{ s.u.}}{=} \sum_{i=1}^{n} \mathrm{Var}(Y_i) = n \cdot \mathrm{Var}(Y_1) = \frac{n(1-p)}{p^2}$$

Aufgabe 35:

a) $X \sim \mathrm{Poi}(\lambda_1)$, $Y \sim \mathrm{Poi}(\lambda_2)$, s.u.

$$P(X=k \mid X+Y=n) = \frac{P(X=k,\ X+Y=n)}{P(X+Y=n)} = \frac{P(X=k,\ Y=n-k)}{P(X+Y=n)}$$

$$\overset{\text{s.u.}}{=} \frac{P(X=k)\,P(Y=n-k)}{P(X+Y=n)} = \frac{e^{-\lambda_1}\cdot\frac{\lambda_1^k}{k!}\cdot e^{-\lambda_2}\frac{\lambda_2^{n-k}}{(n-k)!}}{e^{-\lambda_1-\lambda_2}\frac{(\lambda_1+\lambda_2)^n}{n!}}$$

$$= \binom{n}{k}\frac{\lambda_1^k\lambda_2^{n-k}}{(\lambda_1+\lambda_2)^n} = \binom{n}{k}\Big(\underbrace{\frac{\lambda_1}{\lambda_1+\lambda_2}}_{p}\Big)^k\Big(\frac{\lambda_2}{\lambda_1+\lambda_2}\Big)^{n-k}$$

$$= \binom{n}{k} p^k (1-p)^{n-k}$$

$$\Rightarrow \quad P^{X \mid X+Y=n} = \mathrm{Bin}\Big(n, \frac{\lambda_1}{\lambda_1+\lambda_2}\Big)$$

$$E[X \mid X+Y=n] = n\frac{\lambda_1}{\lambda_1+\lambda_2}$$

b) $X, Y \sim \mathrm{Bin}(n,p)$ i.i.d.

$P(X=k \mid X+Y=m)$

Für $k \leq \min(n,m)$, $\quad$ [sonst: $P(X=k \mid X+Y=m) = 0$]

$$P(X=k \mid X+Y=m) = \frac{P(X=k,\ X+Y=m)}{P(X+Y=m)} = \frac{P(X=k,\ Y=m-k)}{P(X+Y=m)}$$

$$\overset{\text{s.u.}}{=} \frac{P(X=k)\cdot P(Y=m-k)}{P(X+Y=m)} = \frac{\binom{n}{k}p^k(1-p)^{n-k}\cdot\binom{n}{m-k}p^{m-k}(1-p)^{n-m+k}}{\binom{2n}{m}p^m(1-p)^{2n-m}}$$

$$= \frac{\binom{n}{k}\binom{n}{m-k}}{\binom{2n}{m}} \qquad \text{(Hypergeometrische Verteilung)}$$

$E[X \mid X+Y=m]$

$$X \mid X+Y=m \ \hat{=}\ X_1 + X_2 + \dots + X_m$$

$X_1 = 1$: im 1. Zug schw. Kugel
$\quad = 0$: — " — rote — " —

$$\Rightarrow E[X \mid X+Y=m] = E[X_1] + \dots + E[X_m] = \frac{n}{2n} + E[X_2] + \dots + E[X_m]$$

$$= \frac{n}{2n} + E[X_2 \mid X_1=1]\,P(X_1=1) + E[X_2 \mid X_1=0]\,P(X_1=0) + E[X_3] + \dots + E[X_m]$$

$$= \frac{n}{2n} + \frac{1}{2}\frac{n-1}{2n-1} + \frac{1}{2}\frac{n}{2n-1} + \dots$$

$$= \frac{n}{2n} + \frac{n}{2n} + \dots$$

Ind. $= \dfrac{m}{2}$

alternativ: $Y_1 = 1 \;\hat{=}\;$ im 1. Zug rote Kugel
$Y_1 = 0 \;\hat{=}\;$ —— " —— schw. Kugel

$$E[X \mid X+Y=m] = E[Y_1] + \dots + E[Y_m] \overset{!}{=} E[X_1] + \dots + E[X_m]$$

$$= \frac{1}{2} + \dots + \frac{1}{2} = \frac{m}{2}$$

$$E[X] = \sum_{m=0}^{2n} E[X \mid X+Y=m] \cdot P(X+Y=m) = E\big[\,E[X \mid X+Y=m]\,\big] = \dots = n\cdot p$$

Aufgabe 37:

N diskr. ZV, Träger $\mathbb{N}_0$

$X_1, X_2, \dots$ i.i.d.

a) $$E\Big[\sum_{i=1}^{N} X_i\Big] = E\Big[\underbrace{E\Big[\sum_{i=1}^{N} X_i \mid N\Big]}_{(*)}\Big]$$

$$(*)\quad E\Big[\sum_{i=1}^{n} X_i \mid N=n\Big] \overset{\text{s.u.}}{=} E\Big[\sum_{i=1}^{n} X_i\Big] = \sum_{i=1}^{n} E[X_i] = n \cdot E[X_1] \qquad \text{da } X_i \text{ i.i.d.}$$

$$\rightsquigarrow\; E\Big[\sum_{i=1}^{N} X_i \mid N\Big] = N\, E[X_1]$$

$$\Rightarrow\; E\Big[\sum_{i=1}^{N} X_i\Big] = E\big[N\, E[X_1]\big] = E[N] \cdot E[X_1]$$

b) $$\operatorname{Var}\Big(\sum_{i=1}^{N} X_i\Big) = \underbrace{E\Big[\Big(\sum_{i=1}^{N} X_i\Big)^2\Big]}_{(**)} - \Big(E\Big[\sum_{i=1}^{N} X_i\Big]\Big)^2$$

$$\operatorname{Var}(Z) = E(Z^2) - (E(Z))^2$$
$$\Rightarrow E[Z^2] = \operatorname{Var}(Z) + (E[Z])^2$$

$$(**):\quad E\Big[\Big(\sum_{i=1}^{N} X_i\Big)^2\Big] = E\Big[\underbrace{E\Big[\Big(\sum_{i=1}^{N} X_i\Big)^2 \mid N\Big]}_{(***)}\Big]$$

$$(***):\quad E\Big[\Big(\sum_{i=1}^{N} X_i\Big)^2 \mid N\Big] = E\Big[\Big(\sum_{i=1}^{n} X_i\Big)^2 \mid N=n\Big] = E\Big[\Big(\sum_{i=1}^{n} X_i\Big)^2\Big] = \operatorname{Var}\Big(\sum_{i=1}^{n} X_i\Big) + \Big(E\Big[\sum_{i=1}^{n} X_i\Big]\Big)^2$$

$$\underset{\text{s.u.}}{=} \sum_{i=1}^{n} \operatorname{Var}(X_i) + \Big(\sum_{i=1}^{n} E(X_i)\Big)^2 = n \cdot \operatorname{Var}(X_1) + n^2\, E[X_1]^2$$

$$\Rightarrow\; E\Big[\Big(\sum_{i=1}^{N} X_i\Big)^2 \mid N\Big] = N \cdot \operatorname{Var}(X_1) + N^2\, E[X_1]^2$$

$$\Rightarrow (**):\quad E\Big[\Big(\sum_{i=1}^{N} X_i\Big)^2\Big] = E\big[N \cdot \operatorname{Var}(X_1) + N^2 E[X_1]^2\big] = E[N] \cdot \operatorname{Var}(X_1) + E[N^2] \cdot E[X_1]^2$$

$$\Rightarrow \operatorname{Var}\Big(\sum_{i=1}^{N} X_i\Big) = E[N] \cdot \operatorname{Var}(X_1) + E[N^2]\, E[X_1]^2 - \Big(E\Big[\sum_{i=1}^{N} X_i\Big]\Big)^2$$

$$= E[N] \cdot \operatorname{Var}(X_1) + E[X_1]^2 \operatorname{Var}(N)$$

Aufgabe 36:

$$f_{(X,Y)}(x,y) = 6xy\,(2-x-y)\ \mathbb{1}_{(0,1)^2}(x,y)$$

$$f_{X|Y}(x|y) = \frac{f_{(X,Y)}(x,y)}{f_Y(y)}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{(X,Y)}(x,y)\,dx = \int_0^1 6xy(2-x-y)\,\mathbb{1}_{(0,1)}(y)\,dx = \int_0^1 (12xy - 6x^2y - 6xy^2)\,\mathbb{1}_{\dots}\,dx$$

$$= \left[6x^2y - 2x^3y - 3x^2y^2\right]_0^1 = (6y - 2y - 3y^2) = (4y - 3y^2)\cdot \mathbb{1}_{(0,1)}(y)$$

$$\Rightarrow f_{X|Y}(x|y) = \frac{6xy(2-x-y)\ \mathbb{1}_{(0,1)^2}(x,y)}{y(4-3y)\ \mathbb{1}_{(0,1)}(x,y)} = \frac{6x(2-x-y)}{4-3y}\ \mathbb{1}_{(0,1)^2}(x,y)$$

$$E[X\,|\,Y=y] = \int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\,dx = \int_{-\infty}^{\infty} x\cdot\frac{6x(2-x-y)}{4-3y}\ \mathbb{1}_{(0,1)^2}(x,y)\,dx$$

$$= \int_0^1 \frac{6x^2(2-x-y)}{4-3y}\ \mathbb{1}_{(0,1)}(y)\,dx = \int_0^1 \frac{12x^2 - 6x^3 - 6x^2y}{4-3y}\ \mathbb{1}_{(0,1)}(y)\,dx$$

$$= \frac{1}{4-3y}\left[4x^3 - \frac{3}{2}x^4 - 2x^3y\right]_0^1 \mathbb{1}_{(0,1)}(y)$$

$$= \frac{4 - \frac{3}{2} - 2y}{4-3y}\ \mathbb{1}_{(0,1)}(y) = \frac{\frac{5}{2} - 2y}{4-3y}\ \mathbb{1}_{(0,1)}(y)$$

P-fast sicher: $P(\lim_{n\to\infty} X_n = X) = 1$

Im W'raum konvergiert die Folge $(X_n)_{n\in\mathbb{N}}$ mit W'keit 1 gegen X – alle Ereignisse, für die die Folge nicht konvergiert haben ~~zusammen~~ die W'keit 0.

P-stochastisch: $\lim_{n\to\infty} P(|X_n - X| > \varepsilon) = 0 \quad \forall \varepsilon > 0$

$X \sim R(0,1)$

$$P(\max\{X,1-X\} \le x) = P(\max\{X,1-X\} \le x \mid X \le 1-X) \cdot P(X \le 1-X)$$

(Satz v. d. tot. W'keit)

$$+ P(\max\{X,1-X\} \le x \mid X > 1-X) \cdot P(X > 1-X)$$

$$\left(\begin{aligned} &= P(1-X \le x) \cdot P(X \le \tfrac{1}{2}) + P(X \le x) \cdot P(X > \tfrac{1}{2}) \\ &= P(X \ge 1-x) \cdot P(X \le \tfrac{1}{2}) + P(X \le x) \cdot P(X > \tfrac{1}{2}) \\ &= (1 - P(X \le 1-x)) \cdot \tfrac{1}{2} + P(X \le x) \cdot \tfrac{1}{2} \\ &= (1-(1-x))\tfrac{1}{2} + x \cdot \tfrac{1}{2} = x \end{aligned} \right)$$

$$= \frac{P(1-X \le x)}{P(X \le 1-x)} \cdot P(X \le 1-x) + \frac{P(X \le x)}{P(X > 1-x)} P(X > 1-x)$$

$$= P(X \ge \tfrac{1}{2}) + P(X \le x) = 1 - P(X \le \tfrac{1}{2}) + P(X \le x)$$

$$= 1 - \tfrac{1}{2} + x = x + \tfrac{1}{2}$$

$X \sim R(0,1)$

$$P(\max\{X,1-X\} \le x) = P(\max\{X,1-X\} \le x \mid X \le 1-X)\, P(X \le 1-X)$$

$$+ P(\max\{X,1-X\} \le x \mid X > 1-X)\, P(X > 1-X)$$

$$= P(1-X \le x \mid X \le \tfrac{1}{2})\, P(X \le \tfrac{1}{2}) + P(X \le x \mid X > \tfrac{1}{2}) \cdot P(X > \tfrac{1}{2})$$

Nebenrechnung: $P(X \le x \mid X > \frac{1}{2}) = \frac{P(X \le x,\ X > \frac{1}{2})}{P(X > \frac{1}{2})} = 2 \cdot \begin{cases} 0, & x \le \frac{1}{2} \\ x - \frac{1}{2}, & x > \frac{1}{2} \end{cases}$

$\qquad \overset{||}{2x-1}$ für $\frac{1}{2} < x \le 1$

also $X \mid X > \frac{1}{2} \sim R(\frac{1}{2}, 1)$

analog: $X \mid X \le \frac{1}{2} \sim R(0, \frac{1}{2})$

$$= (1 - P(X \le 1-x \mid X \le \tfrac{1}{2}))\, P(X \le \tfrac{1}{2}) + P(X \le x \mid X > \tfrac{1}{2}) \cdot P(X > \tfrac{1}{2})$$

$$= (1 - 2(1-x)) \cdot \tfrac{1}{2} + (2x-1) \cdot \tfrac{1}{2}$$

$$= \tfrac{1}{2} - (1-x) + x - \tfrac{1}{2} = 2x - 1 \qquad \text{für } \tfrac{1}{2} \le x \le 1$$

$$\Rightarrow \max\{X, 1-X\} \sim R(\tfrac{1}{2}, 1)$$

Aufgabe 38:

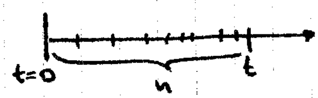
$$N(t) \sim Poi(\lambda t)$$

$$A(t) = \sum_{i=1}^{N(t)} A_i \, e^{-\alpha(t-S_i)} \qquad \text{mit } A_i \text{ i.i.d.}$$

$$E[A(t)] = \sum_{n=0}^{\infty} E[A(t) \mid N(t)=n] \cdot P(N(t)=n)$$

$$= \sum_{n=0}^{\infty} E[A(t) \mid N(t)=n] \cdot e^{-\lambda t} \frac{(\lambda t)^n}{n!}$$

$$N(t) = n$$



$\underbrace{\text{Zwischenankunftszeiten}}_{}$ (unsortiert) können für $N(t)=n$ bel. fest(!) als gleichverteilt angenommen werden auf $(0,t)$ [da Exp-Vert. gedächtnislos]

$\Rightarrow$ Für $N(t)=n$ hat $A(t)$ die gleiche Verteilung wie

$$\sum_{j=1}^{n} A_j \, e^{-\alpha(t-Y_j)} \qquad \text{mit } Y_j \sim R(0,t)$$

$$\Rightarrow E[A(t) \mid N(t)=n] = E\left[\sum_{j=1}^{n} A_j \, e^{-\alpha(t-Y_j)}\right] \underset{E\text{-linear}}{\overset{\text{s.u.}}{=}} \sum_{j=1}^{n} E[A_j] \cdot E[e^{-\alpha(t-Y_j)}]$$

$$= n \cdot E[A_1] \, E[e^{-\alpha(t-Y_1)}] \qquad \text{da } A_j, Y_j \text{ i.i.d.}$$

$$E[e^{-\alpha(t-Y_1)}] = \int_0^t \frac{1}{t} \cdot e^{-\alpha(t-y)} \, dy = \frac{e^{-\alpha t}}{\alpha t} e^{\alpha y} \Big|_0^t = \frac{1-e^{-\alpha t}}{\alpha t}$$

$$\Rightarrow E[A(t) \mid N(t)=n] = n \cdot E[A_1] \cdot \frac{1-e^{-\alpha t}}{\alpha t}$$

$$E[A(t)] = E[E[A(t) \mid N(t)]] = E\left[N(t) \, E[A_1] \cdot \frac{1-e^{-\alpha t}}{\alpha t}\right]$$

$$= E[N(t)] \cdot E[A_1] \cdot \frac{1-e^{-\alpha t}}{\alpha t}$$

$$= \frac{\lambda}{\alpha} \left(1 - e^{-\alpha t}\right) E[A_1]$$

Aufgabe 39:

$$P(|X-E[X]|>\varepsilon) \leq \frac{Var(X)}{\varepsilon^2}$$

z.z. $\frac{1}{n}\sum_{i=1}^{n}(X_i - E[X_i]) \longrightarrow 0$ P-stoch. $(n\to\infty)$ [WLLN]

$$P\left(\left|\frac{1}{n}\sum_{i=1}^{n}(X_i - E[X_i])\right| > \varepsilon\right) = P\left(\left|\frac{1}{n}\sum_{i=1}^{n}X_i - E\left[\frac{1}{n}\sum_{i=1}^{n}X_i\right]\right| > \varepsilon\right)$$

$$\leq \frac{Var\left(\frac{1}{n}\sum_{i=1}^{n}X_i\right)}{\varepsilon^2} = \frac{1}{n^2\varepsilon^2}\sum_{i=1}^{n}Var(X_i) \leq \frac{1}{n^2\varepsilon^2}\sum_{i=1}^{n}\max\{V(X_i) \mid i=1,\dots,n\}$$

$$\underset{\substack{\text{O.B.d.A.}\\ \text{sei } \max\{\dots\}\\ =Var(X_1)}}{=} \frac{n}{n^2\varepsilon^2}Var(X_1) = \frac{1}{n\varepsilon^2}Var(X_1) \xrightarrow[n\to\infty]{} 0$$

Aufgabe 41:

$$D_n = \sqrt{\sum_{i=1}^{n}(X_i - Y_i)^2}$$

Def.: $S_n = \frac{1}{n}\sum_{i=1}^{n}(X_i - Y_i)^2$

$$E\left[\sqrt{S_n}\right] = E\left[\sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i-Y_i)^2}\right] = n^{-\frac{1}{2}}\,E\left[\sqrt{\sum_{i=1}^{n}(X_i-Y_i)^2}\right] = n^{-1/2}\,E[D_n].$$

Def.: $Z_i := (X_i - Y_i)^2$

$$E[Z_i] = E[(X_i-Y_i)^2] = E[X_i^2 - 2X_iY_i + Y_i^2] = E[X_i^2] - 2E[X_i]E[Y_i] + E[Y_i^2]$$

$$= 2\,Var(X_i) = \frac{2}{12} = \frac{1}{6}$$

$$Var(Z_i) = Var((X_i-Y_i)^2) = Var(X_i^2 - 2X_iY_i + Y_i^2) = E[(X_i^2-2X_iY_i+Y_i^2)^2] - \left(E[X_i^2-2X_iY_i+Y_i^2]\right)^2$$

$$= \dots = 2\cdot E[X_i^4] - 8\,E[X_i^3]\,E[X_i] + 6E[X_i^2] - \frac{1}{36}$$

$E[X_i^2] = \frac{1}{3}$

$E[X_i^3] = \int_{-\infty}^{\infty} x^3\, \mathbb{1}_{(0,1)}(x)\,dx = \int_0^1 x^3\,dx = \frac{1}{4}x^4\Big|_0^1 = \frac{1}{4}$

$E[X_i^4] = \frac{1}{5}$

$\Rightarrow Var(Z_i) \leq M < \infty$

$\Rightarrow$ WLLN-Bed. erfüllt

$\Rightarrow \frac{1}{n}\sum_{i=1}^{n}(Z_i - E[Z_i]) \xrightarrow[n\to\infty]{} 0$ P-stoch.

$$\left(\frac{1}{n}\sum_{i=1}^{n}Z_i - \frac{n}{n}E[Z_i]\right) \to 0 \Rightarrow \frac{1}{n}\sum_{i=1}^{n}Z_i - \frac{1}{6} \to 0 \Rightarrow \frac{1}{n}\sum_{i=1}^{n}Z_i \to \frac{1}{6}$$

$$\Rightarrow \sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i-Y_i)^2} \longrightarrow \sqrt{\frac{1}{6}}$$

Aufgabe 39:

$\{X_n\}_{n\in\mathbb{N}}$ paarw. unkorreliert, $Var(X_n) \leq M < \infty$.

o.B.d.A. ist $E[X_i] = 0$ (sonst transf. $Y := X - E[X]$, so dass

$$E[Y] = E[X - E[X]] = E[X] - E[X] = 0)$$

Setze $\bar{X_n} = \frac{1}{n}\sum_{i=1}^{n} X_i$.

Dann ist $Var[\bar{X}_n] = Var\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n^2}\sum_{i=1}^{n} Var[X_i]$, da X_i paarw. unkor.

$$\leq \frac{1}{n^2}\cdot n\cdot M = \frac{M}{n}$$

Dann gilt nach der Tschebyscheff-Ungl.

$$P(|\bar{X}_n| > \varepsilon) \leq \frac{Var(\bar{X}_n)}{\varepsilon^2} \leq \frac{M}{n\cdot\varepsilon^2}$$

und damit

$$\lim_{n\to\infty} P(|\bar{X}_n| > \varepsilon) = \lim_{n\to\infty} P\left(\frac{1}{n}\sum(X_i - E[X_i]) > \varepsilon\right)$$

$$\leq \lim_{n\to\infty} \frac{M}{n\cdot\varepsilon^2} = 0 \quad \text{für } M<\infty,\ \varepsilon>0$$

$\Rightarrow \frac{1}{n}\sum(X_i - E[X_i]) \xrightarrow[(n\to\infty)]{} 0$ P-stoch. $\square$

Aufgabe 40:

$X_2, X_3, \dots$ s.u. ZV

$$P(X_n = n) = P(X_n = -n) = \frac{1}{2n\log(n)} \quad , \quad P(X=0) = 1 - \frac{1}{n\log(n)}$$

z.z.: $\frac{1}{n}\sum_{i=1}^{n}(X_i - E[X_i]) \xrightarrow[(n\to\infty)]{} ?$

$$E[X_n] = 0 \Rightarrow E[X_n]^2 = 0$$

$$E[X_n^2] = (-n)^2 \frac{1}{2n\log(n)} + n^2 \frac{1}{2n\log(n)} = \frac{n}{\log(n)} \Rightarrow Var(X_n) = \frac{n}{\log(n)}$$

Def: $S_n = \frac{1}{n}\sum_{i=1}^{n}(X_i - E[X_i]) = \frac{1}{n}\sum_{i=1}^{n} X_i$

$$\lim_{n\to\infty} P(|S_n| > \varepsilon) = \lim_{n\to\infty} P\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i\right| > \varepsilon\right) = \lim_{n\to\infty} P\left(\left|\sum_{i=1}^{n} X_i\right| > n\varepsilon\right) \overset{\text{Tschebyscheff}}{\leq} \lim_{n\to\infty} \frac{1}{(n\varepsilon)^2} Var\left(\sum_{i=1}^{n} X_i\right)$$

$$= \lim_{n\to\infty} \frac{1}{n^2\varepsilon^2}\sum_{i=1}^{n} Var(X_i) = \lim_{n\to\infty} \frac{1}{n^2\varepsilon^2} \cdot \sum_{i=1}^{n} \frac{i}{\log(i)} \leq \lim_{n\to\infty} \frac{1}{n^2\varepsilon^2} \frac{n^2}{\log(n)}$$

$$= \lim_{n\to\infty} \frac{1}{\varepsilon^2 \log(n)} = 0$$

Also $S_n \xrightarrow[(n\to\infty)]{} S$ P-stoch.

$$\sum_{i=2}^{\infty} P(|X_i| \geq i) \geq \sum_{i=2}^{\infty} \frac{1}{n\log(n)} = \infty \qquad \left(\text{Integral-Kriterium} \quad \int \frac{1}{x \ln x}\,dx = \ln\ln x + c\right)$$

$\Rightarrow$ (Borel-Cantelli) $|X_i| \geq i$ tritt mit Wkeit 1 ∞ oft auf!

$\Rightarrow |S_i - S_{i-1}| = |X_i| \geq i$

$\Rightarrow \frac{S_n}{n}$ konvergiert nicht P-fast sicher

Aufgabe 42:

$X_i \mathrel{\hat=}$ Gewicht von Gast i

$E[X_i] = 75 =: \mu$, $Var(X_i) = 49 =: \sigma^2$

$$\sum_{i=1}^{80} X_i \leq 6800 \quad , \quad \sum_{i=81}^{240} X_i \leq 13600 \quad , \quad 2{,}5 \sum_{i=1}^{80} X_i \geq \sum_{i=81}^{240} X_i$$

$S_n = \sum_{i=1}^{n} X_i$ mit $E[S_n] = n\cdot\mu$; $Var(S_n) = n\cdot\sigma^2$

mit $Y_n = \frac{S_n - n\mu}{\sqrt{n\sigma^2}} = \frac{1}{\sigma\sqrt{n}} \cdot \sum_{i=1}^{n}(X_i - \mu) \overset{as.}{\sim} N(0,1)$

$T(y_n) = y_n\sqrt{n\sigma^2} + n\mu$

Trafo-satz

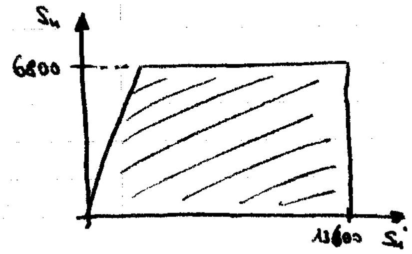
$\Rightarrow S_n \sim N(n\mu, n\sigma^2)$

$$P\left(S_{80} \le 6800,\ S'_{160} \le 13600,\ 2{,}5 \cdot S_{80} \ge S'_{160}\right)$$

$$S'_{160} = \sum_{i=81}^{240} X_i$$

$n_1 = 80$
$n_2 = 160$

$$= \int_0^{6800} \int_0^{\min\{2{,}5x,\ 13600\}} \frac{1}{\sqrt{n_1}\,\sigma\sqrt{2\pi}}\, e^{-\frac{(x-n_1\mu)^2}{2\sigma^2\sqrt{n_1}}} \cdot \frac{1}{\sqrt{n_2}\,\sigma\sqrt{2\pi}}\, e^{-\frac{(y-n_2\mu)^2}{2\sigma^2\sqrt{n_2}}}\, dy\, dx$$



$$P(\ldots) \approx 1 \quad !$$

Aufgabe 43:

$X_1, \ldots, X_n$ i.i.d. $\sim R(0,\theta)$

$$\Rightarrow L(\theta \mid x_1, \ldots, x_n) = f(x_1, \ldots, x_n \mid \theta) = \prod_{i=1}^{n} \frac{1}{\theta} \mathbb{1}_{[0,\theta]}(x_i) = \frac{1}{\theta^n} \prod_{i=1}^{n} \mathbb{1}_{[0,\theta]}(x_i)$$

suche $\max\{L(\theta \mid x_1, \ldots, x_n)\}$

Sei $\theta < \max\{x_i \mid 1 \le i \le n\} \Rightarrow \exists x_i > \theta \Rightarrow \mathbb{1}_{[0,\theta]}(x_i) = 0 \Rightarrow L(\theta \mid x_1, \ldots, x_n) = 0$

Sei $\theta \ge \max\{x_i \mid 1 \le i \le n\} \Rightarrow \forall x_i$ ist $\mathbb{1}_{[0,\theta]}(x_i) = 1 \Rightarrow L(\theta \mid x_1, \ldots, x_n) = \frac{1}{\theta^n}$

maximiere dieses

$\Rightarrow$ minimiere $\theta^n \Rightarrow \hat{\theta} = \max\{x_i \mid 1 \le i \le n\}$

$$P(\max\{x_i \mid 1 \le i \le n\} \le x) = P(X_1 \le x, \ldots, X_n \le x) = \prod_{i=1}^{n} P_{X_i}(x_i \le x) = \left(P(X_1 \le x)\right)^n = \left(\frac{x}{\theta}\right)^n$$

$$\Rightarrow F_{\max\{x_i \mid 1 \le i \le n\}}(x) = \begin{cases} 0 & x < 0 \\ \left(\frac{x}{\theta}\right)^n & 0 \le x \le \theta \\ 1 & x > \theta \end{cases}$$

$$E[\hat{\theta}] = E[\max\{x_i \mid 1 \le i \le n\}] = -\int_{-\infty}^{0} F_{\max}\, dx + \int_0^{\infty} (1 - F_{\max})\, dx = \int_0^{\theta} (1 - F_{\max})\, dx$$

$$= \int_0^{\theta} \left(1 - \left(\frac{x}{\hat{\theta}}\right)^n\right) dx = \hat{\theta} - \frac{1}{n+1} \frac{x^{n+1}}{\hat{\theta}^n}\Big|_0^{\hat{\theta}} = \hat{\theta} - \frac{1}{n+1}\hat{\theta} = \frac{n}{n+1}\hat{\theta}$$

$$\Rightarrow \alpha = \frac{n+1}{n}$$

Aufgabe 45:

$$X \sim N(\theta, \sigma^2)$$

$\theta \sim N(\mu, \tau^2)$ sei die a-priori-Verteilung

$$\Rightarrow f_\theta(\theta) = \frac{1}{\tau\sqrt{2\pi}} \cdot e^{-\frac{(\theta-\mu)^2}{2\tau^2}}$$

$$f_{X|\theta}(x|\theta) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\theta)^2}{2\sigma^2}}$$

Berechne Bayes-Schätzer:

$$f(\theta|x) = \frac{f(x|\theta)\cdot f_\theta(\theta)}{\int f(x|\theta)\cdot f_\theta(\theta)\, d\theta}$$

$$f(x|\theta)\cdot f_\theta(\theta) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\theta)^2}{2\sigma^2}} \cdot \frac{1}{\tau\sqrt{2\pi}} \cdot e^{-\frac{(\theta-\mu)^2}{2\tau^2}}$$

$$= \frac{1}{2\pi\tau\sigma} \cdot \exp\left(\frac{-\tau^2(x^2-2\theta x+\theta^2) - \sigma^2(\theta^2-2\theta\mu+\mu^2)}{2\sigma^2\tau^2}\right)$$

$$= \frac{1}{2\pi\sigma\tau} \exp\left(-\frac{1}{2\sigma^2\tau^2}\left((\tau^2+\sigma^2)\theta^2 - 2\theta(x\tau^2+\mu\sigma^2) + x^2\tau^2+\mu^2\sigma^2\right)\right)$$

$$= \frac{1}{2\pi\tau\sigma} \exp\left(\frac{\tau^2+\sigma^2}{-2\sigma^2\tau^2}\left(\theta^2 + 2\theta\frac{-x\tau^2-\mu\sigma^2}{\tau^2+\sigma^2} + \frac{x^2\tau^2+\mu^2\sigma^2}{\tau^2+\sigma^2}\right)\right)$$

$$= \frac{1}{\tau\sigma 2\pi} \exp\left(-\frac{\left(\theta - \frac{x\tau^2+\mu\sigma^2}{\tau^2+\sigma^2}\right)^2 - \left(\frac{x\tau^2+\mu\sigma^2}{\tau^2+\sigma^2}\right)^2 + \frac{x^2\tau^2+\mu^2\sigma^2}{\tau^2+\sigma^2}}{\frac{2\sigma^2\tau^2}{\tau^2+\sigma^2}}\right)$$

$$\Rightarrow f(\theta|x) \overset{\text{Normierung!}}{=} \frac{1}{\tau\sigma\sqrt{2\pi}} \cdot \exp\left(-\frac{\left(\theta - \frac{x\tau^2+\mu\sigma^2}{\tau^2+\sigma^2}\right)^2}{\frac{2\sigma^2\tau^2}{\tau^2+\sigma^2}}\right)$$

$$\Rightarrow \theta|x \sim N\left(\frac{\tau^2}{\tau^2+\sigma^2}x + \frac{\sigma^2}{\tau^2+\sigma^2}\mu,\ \frac{\sigma^2\tau^2}{\tau^2+\sigma^2}\right)$$