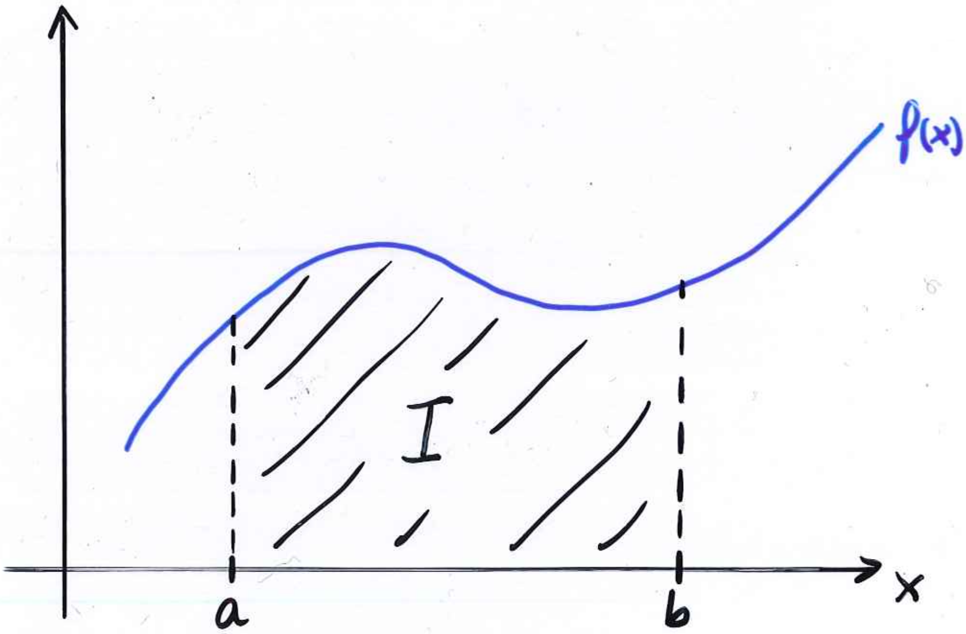
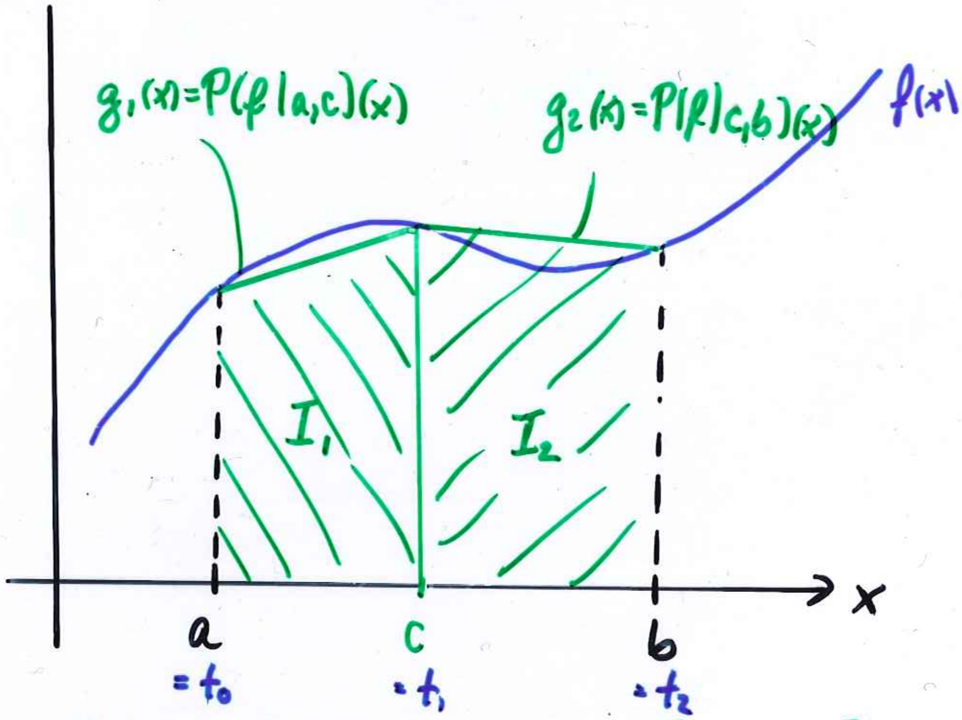
Numerische Integration

Berechne $I = \int_a^b f(x)\,dx$

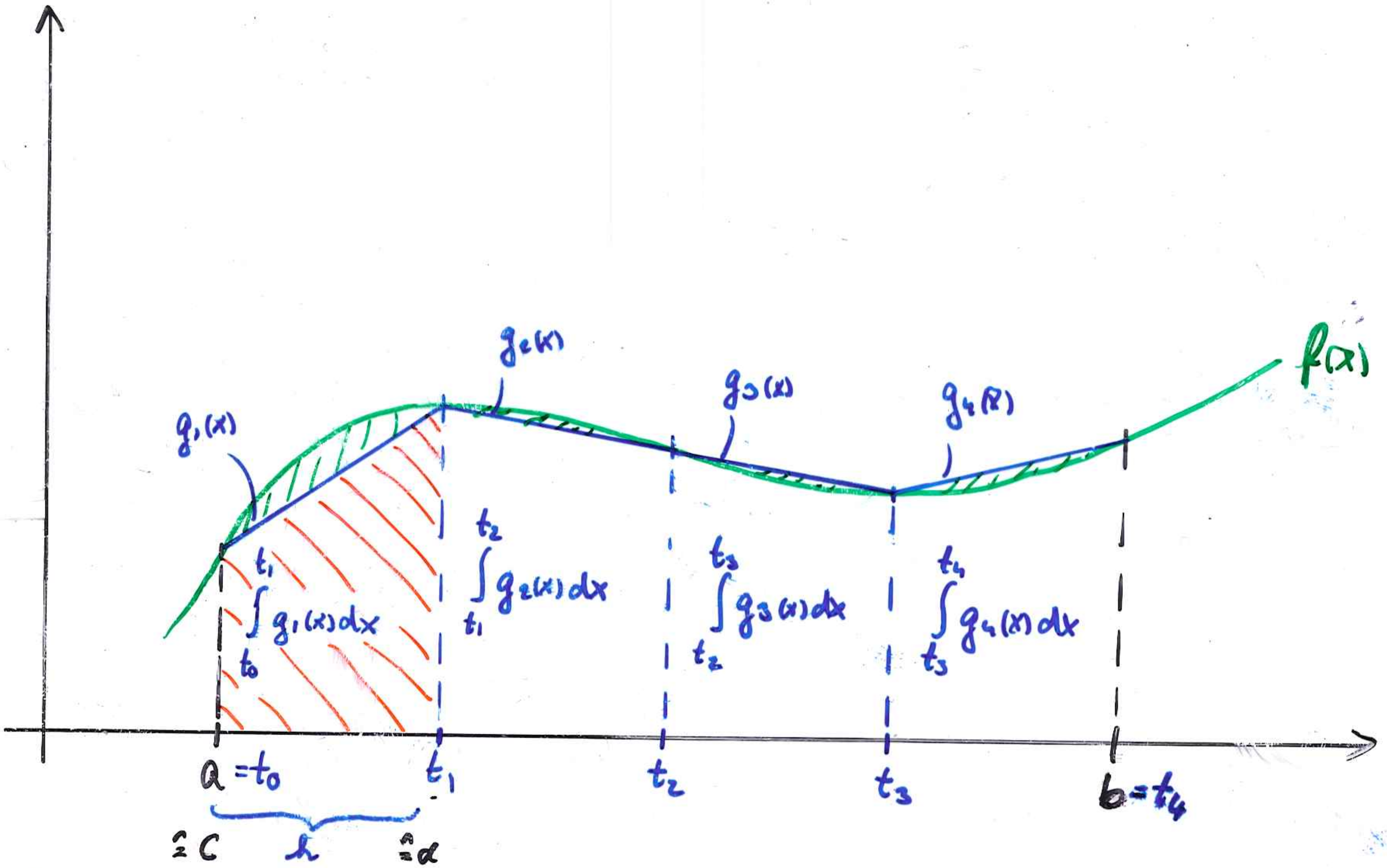


Strategie



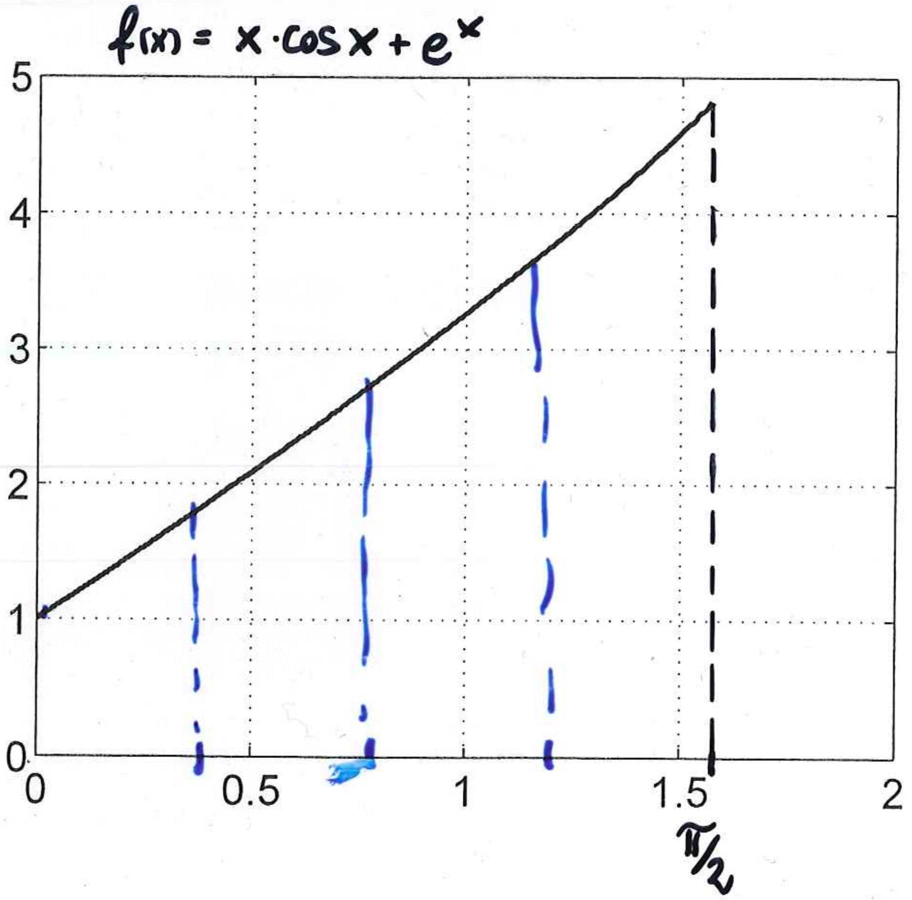
$I_1 = \frac{1}{2}[f(a)+f(c)]\cdot(c-a)$ $\quad$ $I_2 = \frac{1}{2}[f(c)+f(a)]\cdot(b-c)$

Trapezregel: lineare Interpolation

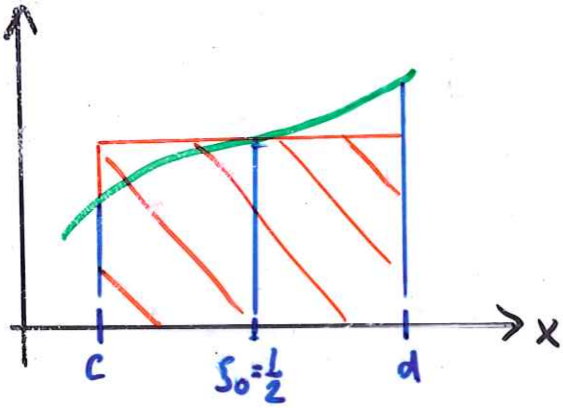


$$\int_{t_0}^{t_1} g_1(x)\,dx = \frac{h}{2}\left[f(t_0) + f(t_1)\right]$$

Beispiel 10.2

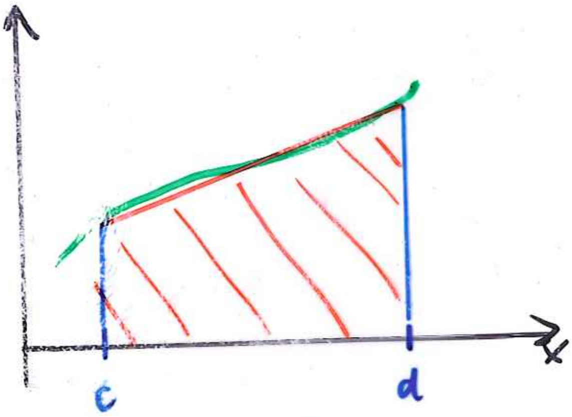


Newton-Cotes



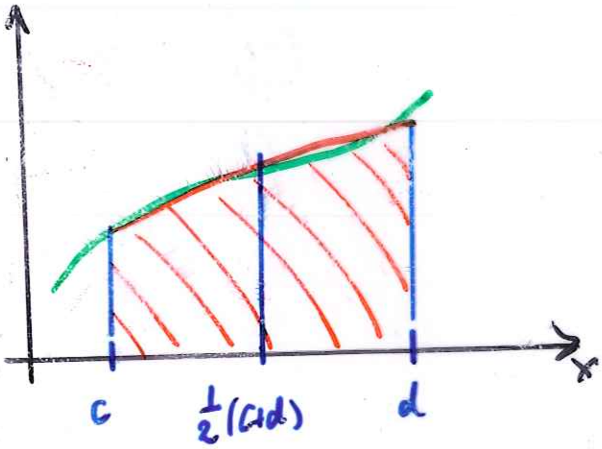
Mittelpunktsregel: $(m=0)$

$$I_0(f) = h \cdot f\left(c + \tfrac{1}{2} h\right)$$



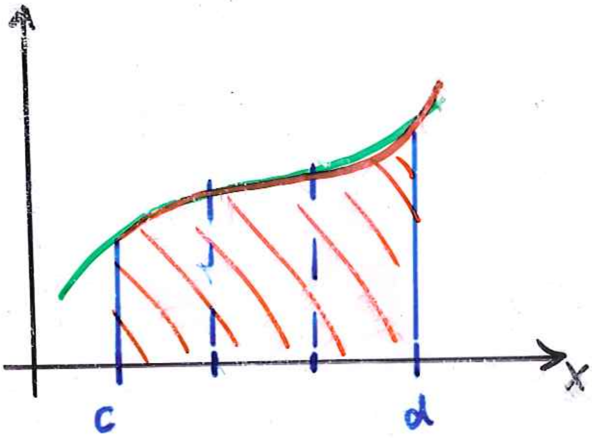
Trapezregel: $(m=1)$

$$I_1(f) = h \left[\tfrac{1}{2} f(c) + \tfrac{1}{2} f(d)\right]$$



Simpson-Regel: $(m=2)$

$$I_2(f) = h \cdot \left[\tfrac{1}{6} f(c) + \tfrac{4}{6} f\left(c + \tfrac{1}{2} h\right) + \tfrac{1}{6} f(d)\right]$$



$\frac{3}{8}$-Regel: $(m=3)$

$$I_3(f) = h \cdot \left[\tfrac{1}{8} f(c) + \tfrac{3}{8} f\left(c + \tfrac{1}{3} h\right) + \tfrac{3}{8} f\left(c + \tfrac{2}{3} h\right) + \tfrac{1}{8} f(d)\right]$$

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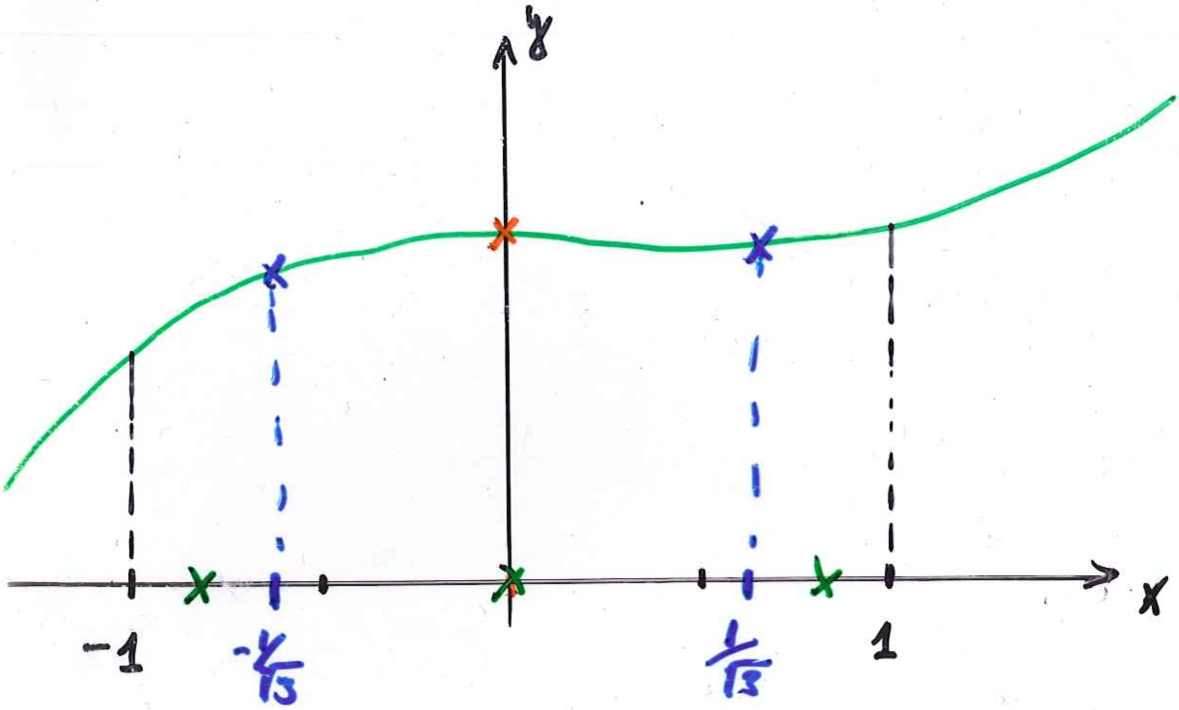


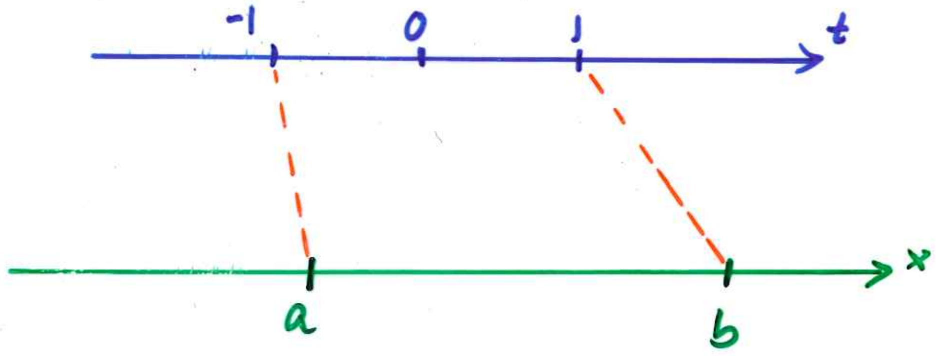
Gauss-Quadratur

Stützstellen x_i und Gewichte w_i auf [-1, 1]

n	x_i	w_i
0	$x_0 = 0$	$w_0 = 2$
1	$x_1 = -x_0 = 1/\sqrt{3}$ = 0,57735...	$w_0 = w_1 = 1$
2	$x_2 = -x_0 = \sqrt{3/5}$ = 0,774596...	$w_0 = w_2 = 5/9$
	$x_1 = 0$	$w_1 = 8/9$
3	$x_3 = -x_0 = 0.8611363116\ldots$	$w_0 = w_3 = 0.3478548451\ldots$
	$x_2 = -x_1 = 0.3399810436\ldots$	$w_1 = w_2 = 0.6521451549\ldots$
4	$x_4 = -x_0 = 0.9061798459\ldots$	$w_0 = w_4 = 0.2369268851\ldots$
	$x_3 = -x_1 = 0.5384693101\ldots$	$w_1 = w_3 = 0.4786286705\ldots$
	$x_2 = 0$	$w_2 = 0.5688888889\ldots$
5	$x_5 = -x_0 = 0.9324695142032\ldots$	$w_0 = w_5 = 0.1713244923792\ldots$
	$x_4 = -x_1 = 0.6612093864663\ldots$	$w_1 = w_4 = 0.3607615730481\ldots$
	$x_3 = -x_2 = 0.2386191860832\ldots$	$w_2 = w_3 = 0.4679139345727\ldots$

$$\int_{-1}^{1} f(x)\,dx = \sum_{i=0}^{m} w_i\, f(x_i)$$

Gauß-Quadratur : $\int_{-1}^{1} g(t)\,dt$ vs. $\int_{a}^{b} f(x)\,dx$



Koordinatentransformation : $x = mt + c$

wenn $x = a$: $t = -1$

$x = b$: $t = +1$

$$\Rightarrow \left.\begin{array}{l} a = -m + c \\ b = m + c \end{array}\right\} \quad \begin{array}{l} m = \frac{b-a}{2} \\ c = \frac{b+a}{2} \end{array}$$

$$\Rightarrow \quad x(t) = \frac{b-a}{2}t + \frac{b+a}{2}$$

Ableitung : $\frac{dx}{dt} = \frac{b-a}{2}$

Einsetzen

$$\int_{a}^{b} f(x)\,dx = \int_{-1}^{1} f\left(\frac{b-a}{2}t + \frac{b+a}{2}\right)\frac{b-a}{2}\,dt$$

Vergleich Newton-Cotes vs. Gauß

	Newton-Cotes	Gauß
Stützstellen	äquidistant	nicht äquidistant (tabuliert)
Gewichte	wechselnde Vorzeichen (bei höherer Ordnung)	positiv
Exaktheitsgrad	m oder m+1	2m+1
mit (n+1) Knoten, exakt vom Grad	$m = \begin{cases} n, & \text{falls } n \text{ ungerade} \\ n+1, & \text{falls } n \text{ gerade} \end{cases}$	$2n+1$