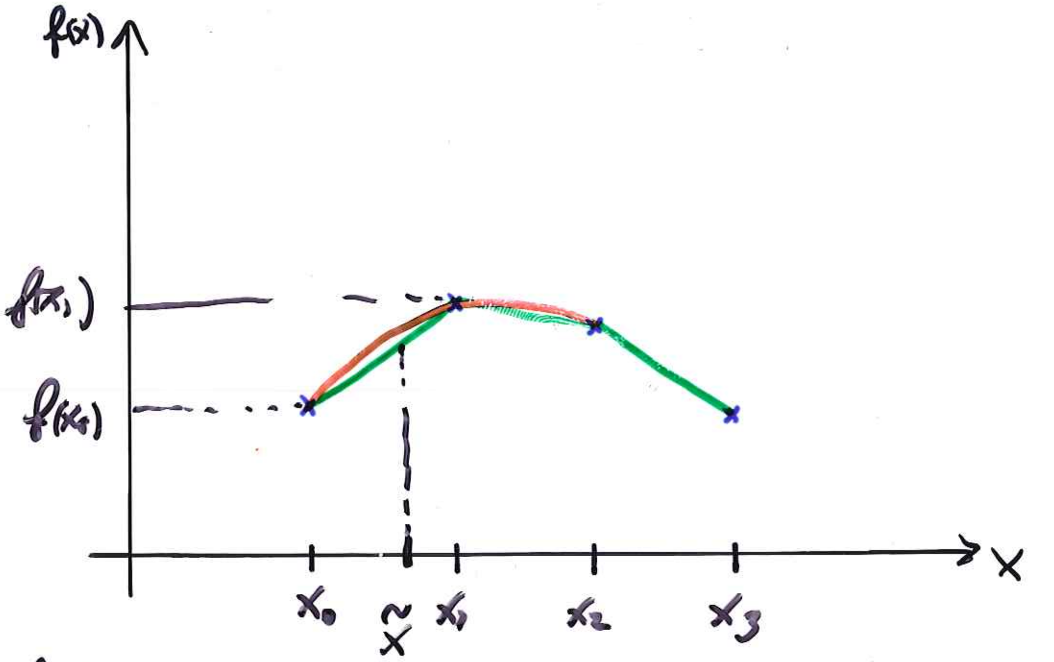
Interpolation

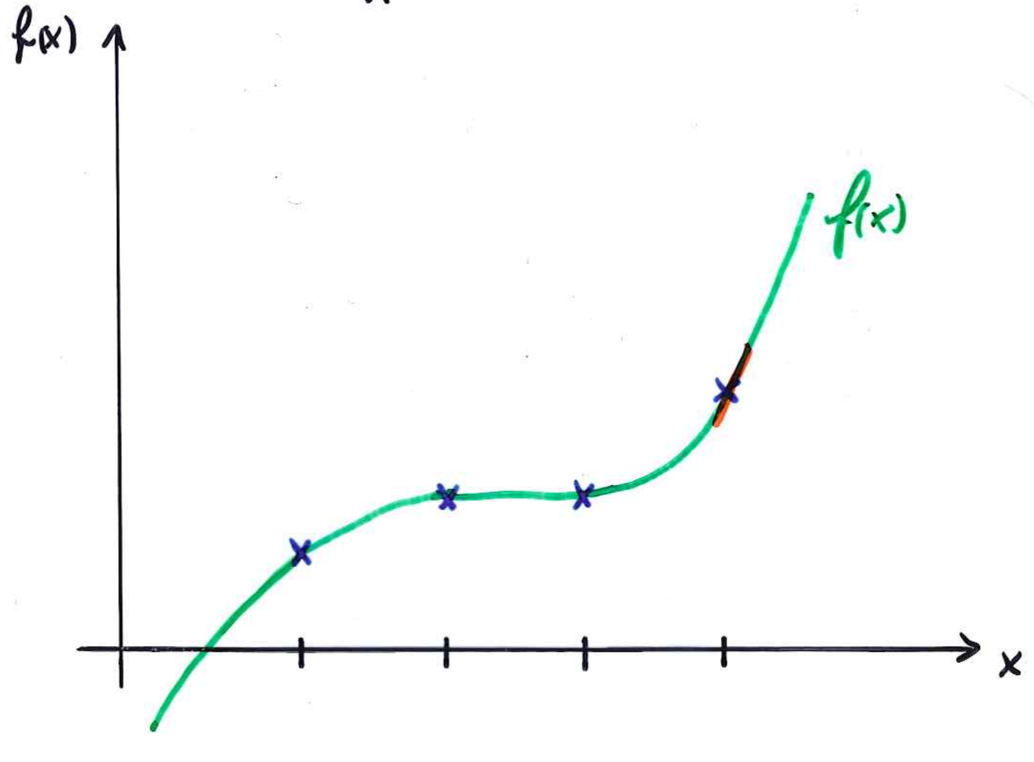
geg.: Stützstellen und zugehörige Daten

$$(x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_n, f(x_n))$$

ges.: Polynom $P_n \in \Pi_n$, so dass

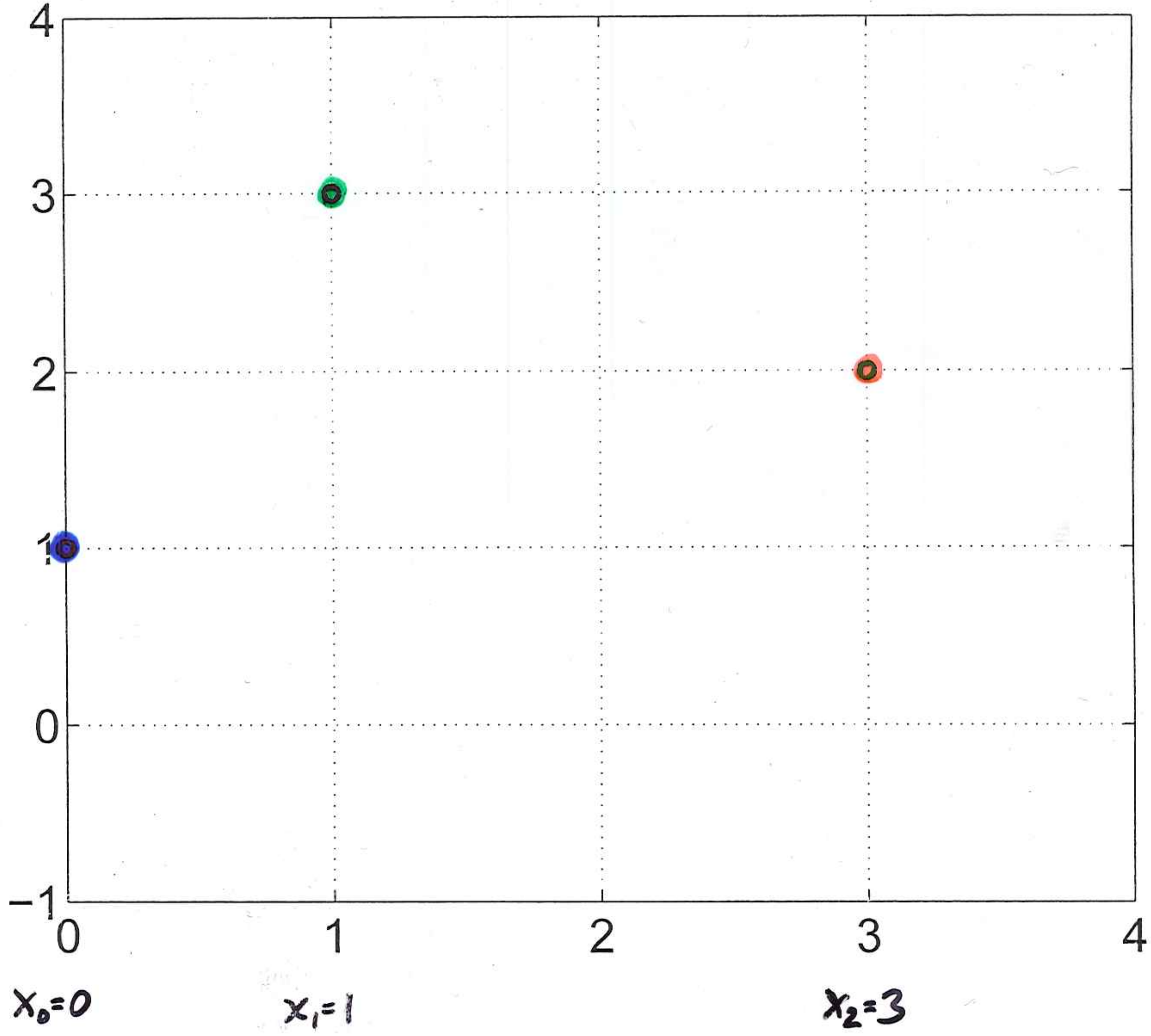
$$P_n(x_j) = f(x_j), \quad j = 0, \ldots, n$$





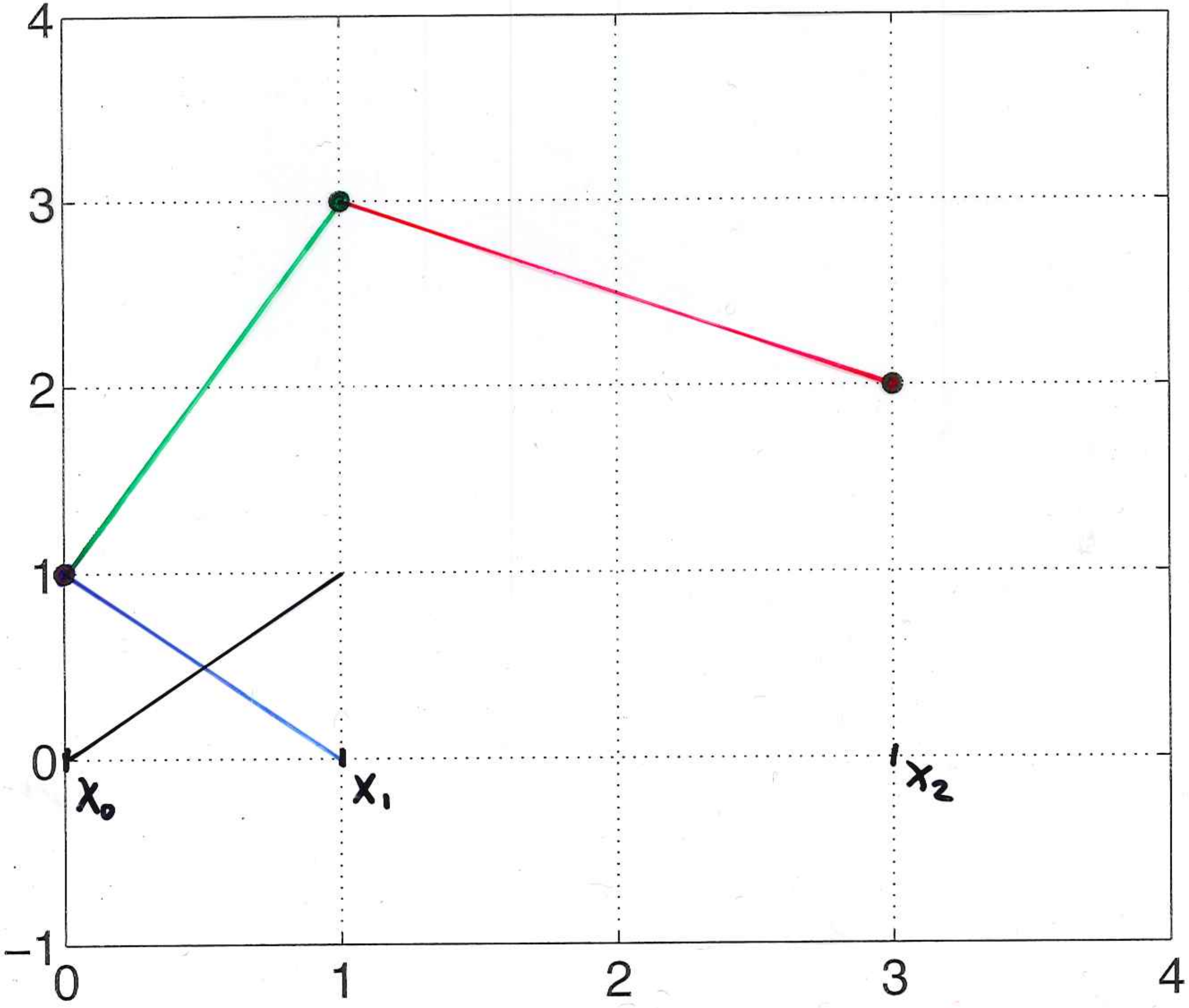
i	0	1	2
x_i	0	1	3
$f(x_i)$	1	3	2

Aufgabe: Finde Interpolationspolynom 2. Grades, das durch $(x_i, f(x_i))$, $i = 0, 1, 2$, geht



N8.2 (2)

i	0	1	2
x_i	0	1	3
$f(x_i)$	1	3	2



$$P(f \mid x_0, x_1)(x) = \underbrace{\frac{x - x_1}{x_0 - x_1}} f(x_0) + \underbrace{\frac{x - x_0}{x_1 - x_0}} f(x_1)$$

$$P(f \mid x_1, x_2)(x) = \frac{x - x_2}{x_1 - x_2} f(x_1) + \frac{x - x_1}{x_2 - x_1} f(x_2)$$

Interpolation:

- Linear: einfach
- Polynom 2. Grades

1. Möglichkeit:

Polynom hat Darstellung: $f(x) = a_0 + a_1 x + a_2 x^2$

$$\left.\begin{aligned} x_0 = 0:&\quad a_0 + a_1 x_0 + a_2 x_0^2 = f(x_0) = 1 \\ &\quad \Rightarrow \quad a_0 = 1 \\ x_1 = 1:&\quad a_0 + a_1 x_1 + a_2 x_1^2 = f(x_1) = 3 \\ x_2 = 3:&\quad a_0 + a_1 x_2 + a_2 x_2^2 = f(x_2) = 2 \end{aligned}\right\} \quad \text{LGS: } Ax = b$$

$$x = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}, \quad A = \underbrace{\begin{bmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{bmatrix}}_{\text{Vandermonde-Matrix}}, \quad b = \begin{bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \end{bmatrix}$$

Problem: - LGS lösen
- Vandermonde-Matrix schlecht konditioniert

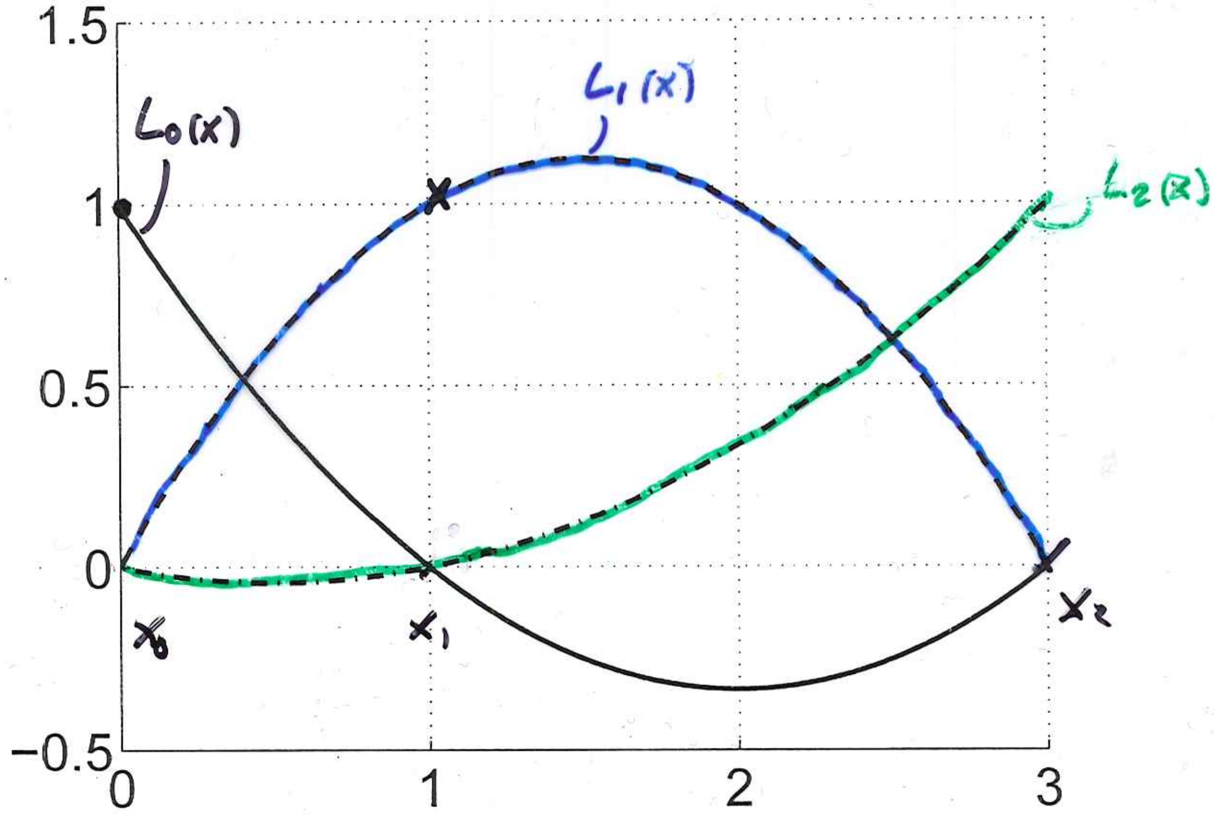
2. Möglichkeit

- Finde Polynom 2. Grades, das durch $(x_0, f(x_0))$ geht {allgemeiner: durch $(x_0, 1)$ geht} und null ist an den anderen Stützstellen.

z.B.

$$L_0(x) = \frac{(x-1)(x-3)}{(x_0-1)(x_0-3)} = \frac{\overbrace{(x-x_1)(x-x_2)}^{\text{garantiert, dass null an } x_1 \text{ und } x_2}}{\underbrace{(x_0-x_1)(x_0-x_2)}_{\text{normiert: 1 an Stelle } x_0}}$$

i	0	1	2
x_i	0	1	3



$$L_0(x) = \frac{(x-1)(x-3)}{(x_0-1)(x_0-3)} = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$$

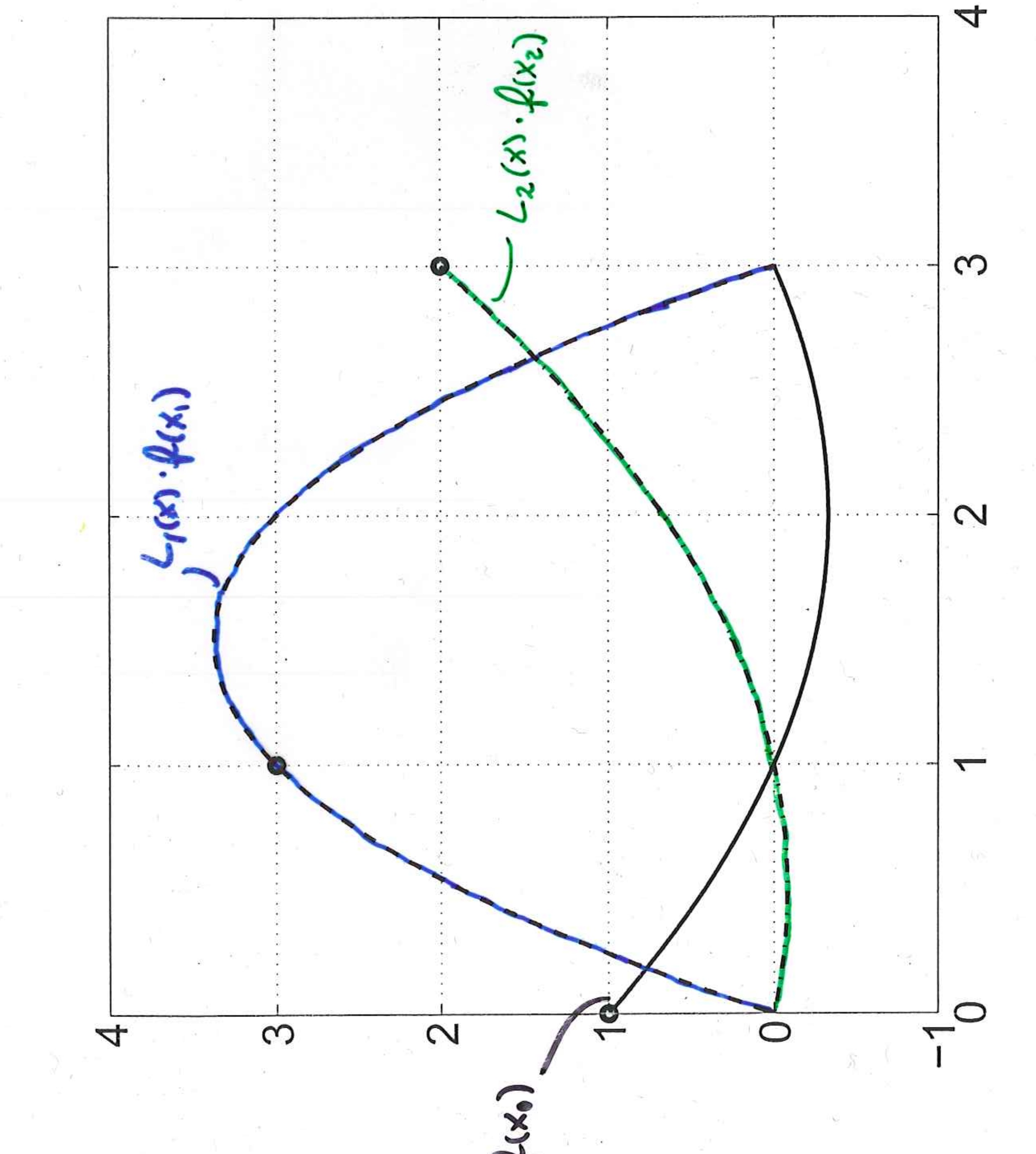
$$L_1(x) = \frac{(x-0)(x-3)}{(x_1-0)(x_1-3)} = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$$

$$L_2(x) = \frac{(x-0)(x-1)}{(x_2-0)(x_2-1)} = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_3-x_1)}$$

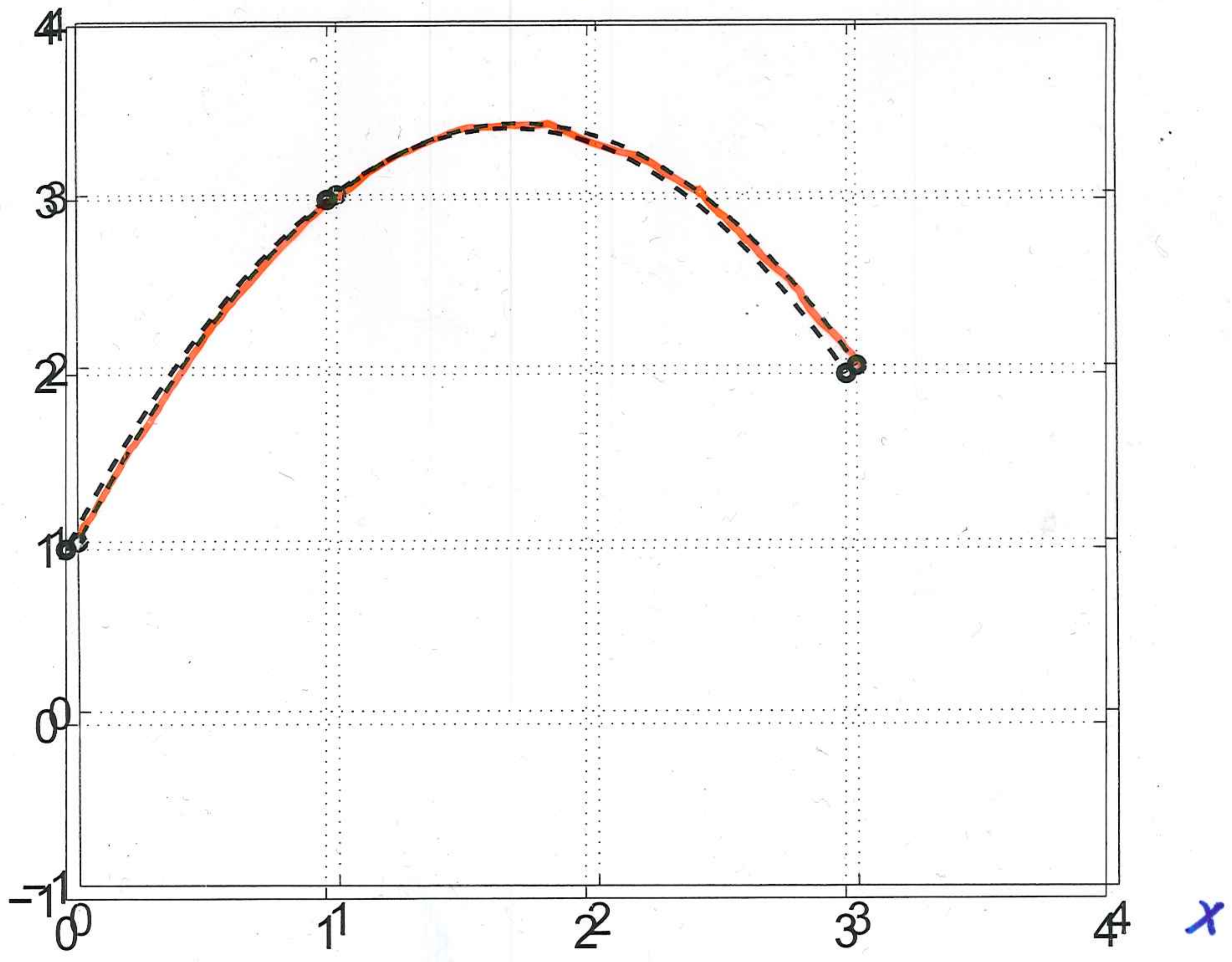
Lagrange-Fundamental-polynome

N8.2 (4)

N8.2 (5)



$$P(f|x_0,x_1,x_2)(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2)$$



N8.2(6)

Lineare Interpolation

$$P(f \mid x_0, x_1)(x) = \underline{\frac{x - x_1}{x_0 - x_1} f(x_0)} + \frac{x - x_0}{x_1 - x_0} f(x_1)$$

$$P(f \mid x_1, x_2)(x) = \frac{x - x_2}{x_1 - x_2} f(x_1) + \underline{\frac{x - x_1}{x_2 - x_1} f(x_2)}$$

Quadratische Interpolation

$$P(f \mid x_0, x_1, x_2)(x) = \frac{(x - x_1)(x - x_2)}{\underline{(x_0 - x_1)}\,\underline{(x_0 - x_2)}} \underline{f(x_0)} + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1)$$

$$+ \frac{(x - x_0)(x - x_1)}{\underline{(x_2 - x_0)}\,\underline{(x_2 - x_1)}} \underline{f(x_2)}$$

$$= \underline{\frac{x - x_2}{x_0 - x_2}} P(f \mid x_0, x_1)(x) + \frac{x - x_0}{\underline{x_2 - x_0}} P(f \mid x_1, x_2)(x)$$

Beispiel 8.7: $n=2$

	$P_{i,0}$	$P_{i,1}$	$P_{i,2}$
$x_0=0$	$P_{0,0}=f(x_0)=1$		
$x_1=1$	$P_{1,0}=f(x_1)=4$	$P_{1,1}=2.5$	
$x_2=2$	$P_{2,0}=f(x_2)=2$	$P_{2,1}=5,0$	$P_{2,2}=3\frac{1}{8}$

geg.: 3 Stützstellen + Funktionswerte

ges.: Funktionswert des quadratischen Interpolationspolynoms an der Stelle $x=0.5$

$$P_{1,1}=P_{1,0}+\frac{x-x_1}{x_1-x_0}\left(P_{1,0}-P_{0,0}\right)=P(f\,|\,x_0,x_1)(x)$$

$$=4+\frac{0.5-1}{1-0}(4-1)=2,5$$

$$P_{2,1}=P_{2,0}+\frac{x-x_2}{x_2-x_1}\left(P_{2,0}-P_{1,0}\right)=P(f\,|\,x_1,x_2)(x)$$

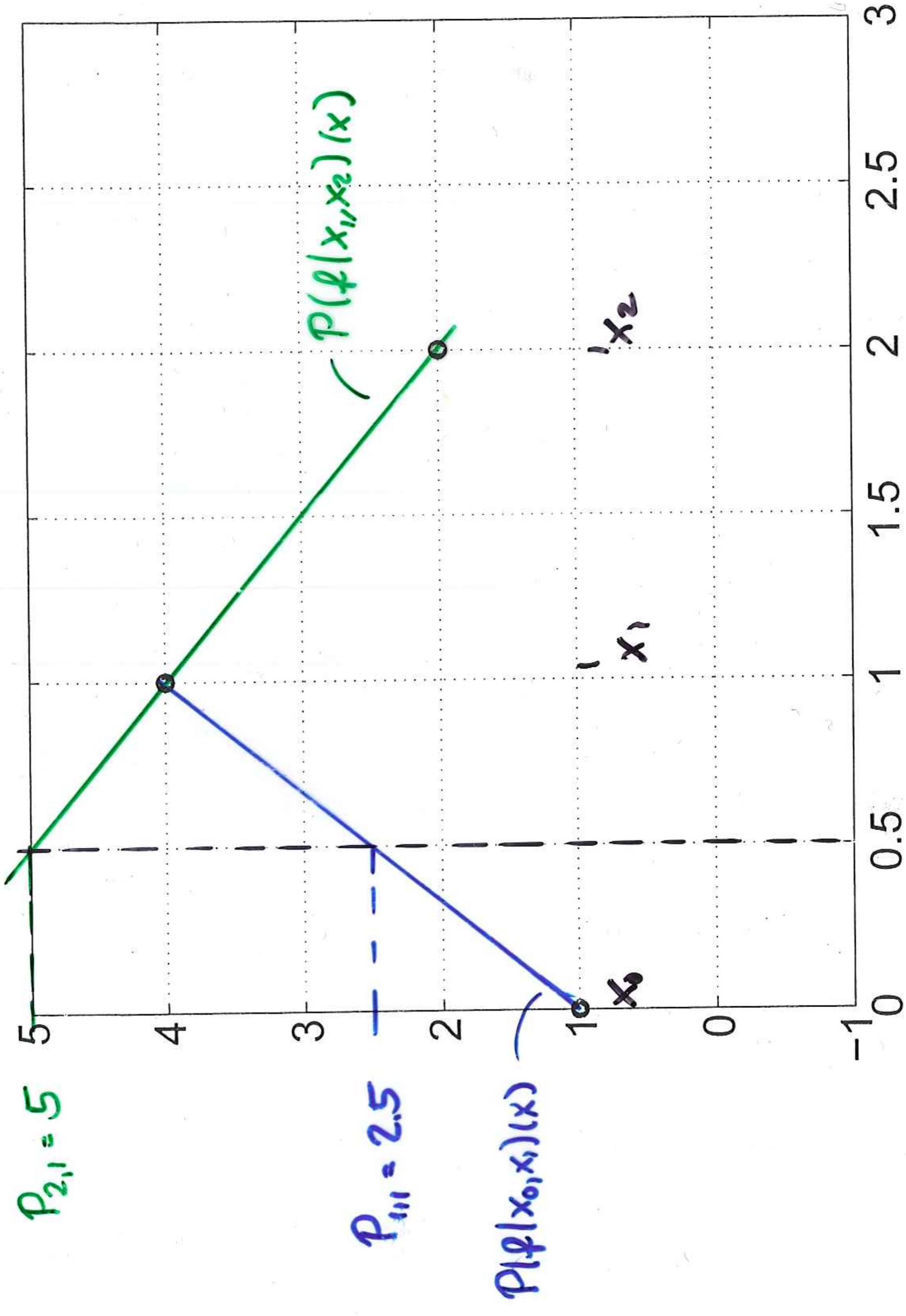
$$=2+\frac{0,5-2}{2-1}(2-4)=5$$

$$P_{2,2}=P_{2,1}+\frac{x-x_2}{x_2-x_0}\left(P_{2,1}-P_{1,1}\right)=P(f\,|\,x_0,x_1,x_2)(x)$$

$$=5+\frac{0.5-2}{2-0}(5-2.5)=3\frac{1}{8}$$

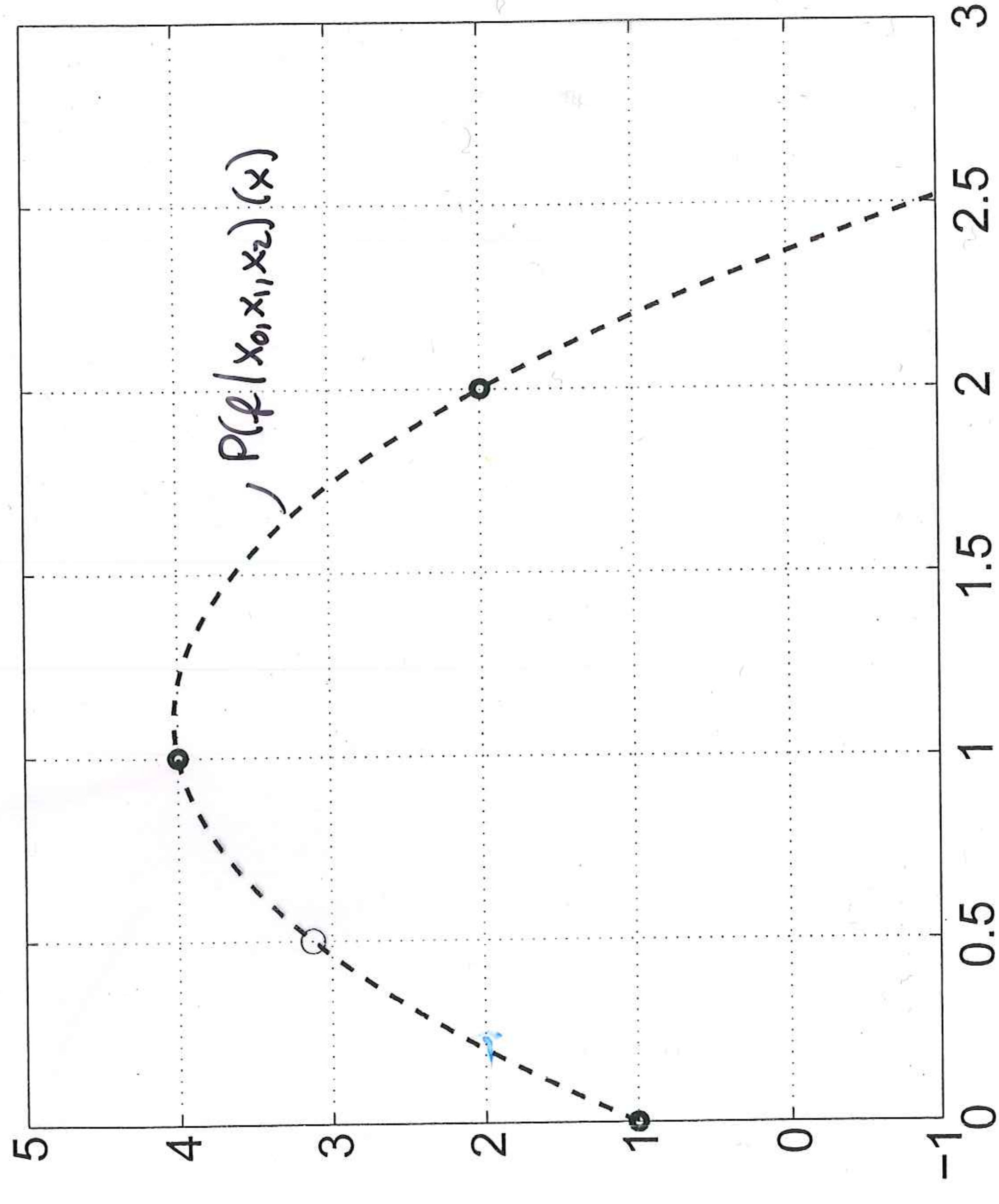
Beispiel: 8.7

N 8.4(2)



N8.4 (3)

Beispiel: 8.7



N8.5

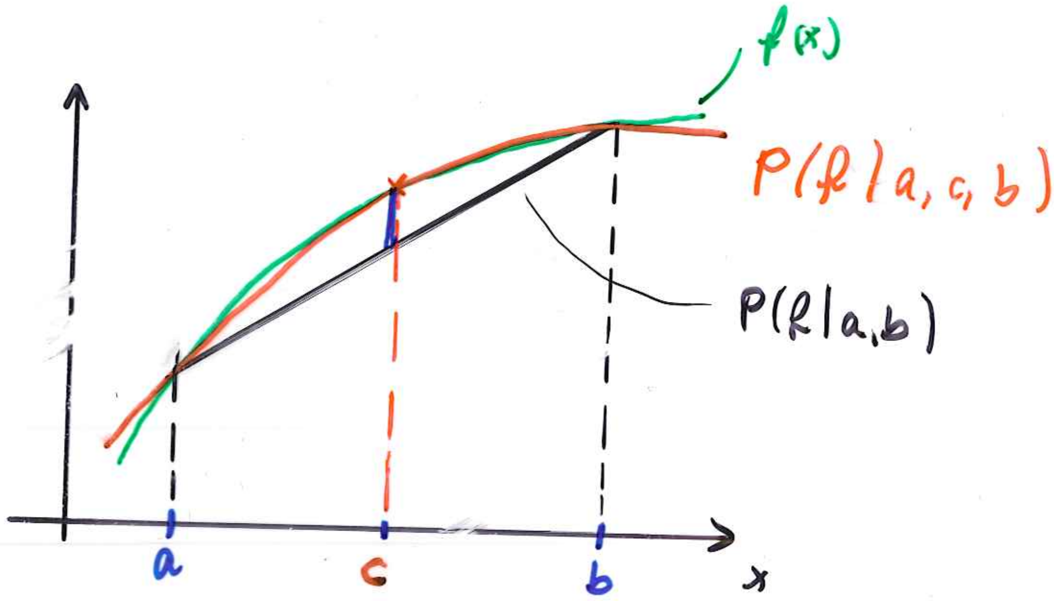
Fehleranalyse: Interpolation

2 Fragestellungen:

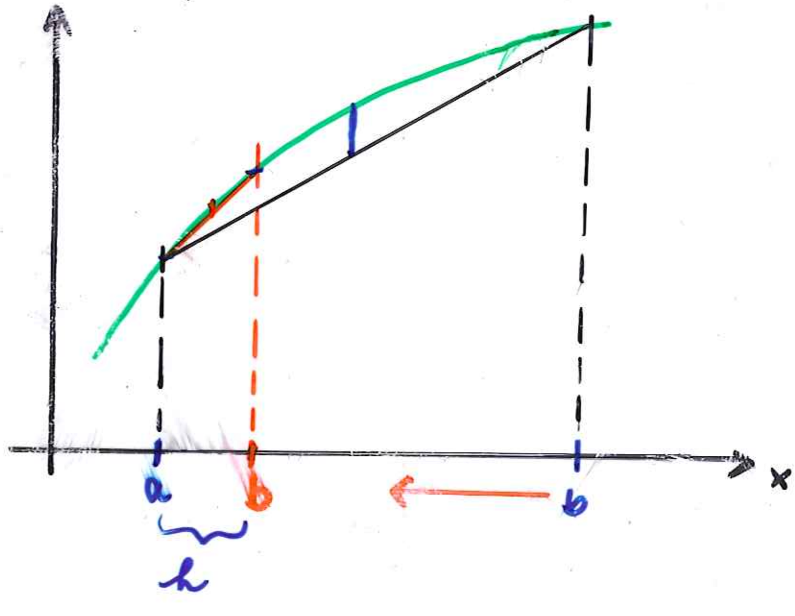
1.) Erhöhung Polynomgrad (bzw. Stützstellenzahl)

2.) Fester Grad

zu 1.)



zu 2.)



Fehleranalyse : allg. Fall

Interpolationsfehler :

$$\left| f(x) - P(f \mid x_0, \dots, x_n)(x) \right| \leq \max_{x \in [a,b]} \left| \omega_{n+1}(x) \right| \max_{x \in [a,b]} \frac{\left| f^{(n+1)}(x) \right|}{(n+1)!}$$

mit

* $P(f \mid x_0, \dots, x_n)(x)$ .... Interpolationspolynom vom Grad n

* $\omega_{n+1}(x) = \prod_{j=0}^{n} (x - x_j) = (x - x_0)(x - x_1) \dots (x - x_n)$ ... Knotenpolynom vom Grad $n+1$

* $f^{(n+1)}(x)$ ... $(n+1)$-te Ableitung von $f(x)$

Fehleranalyse

1) Lineare Interpolation

$$f(x) - P(f \mid x_0, x_1)(x) = (x-x_0)(x-x_1)\,\frac{f''(\xi_1)}{2}$$

mit $\xi_1 \in [x_0, x_1]$

$$\Rightarrow \left| f(x) - P(f \mid x_0, x_1)(x) \right| \leq \max_{x\in[x_0,x_1]} \left|(x-x_0)(x-x_1)\right| \max_{\xi_1\in[x_0,x_1]} \left|\frac{f''(\xi_1)}{2}\right|$$

2) Quadratische Interpolation

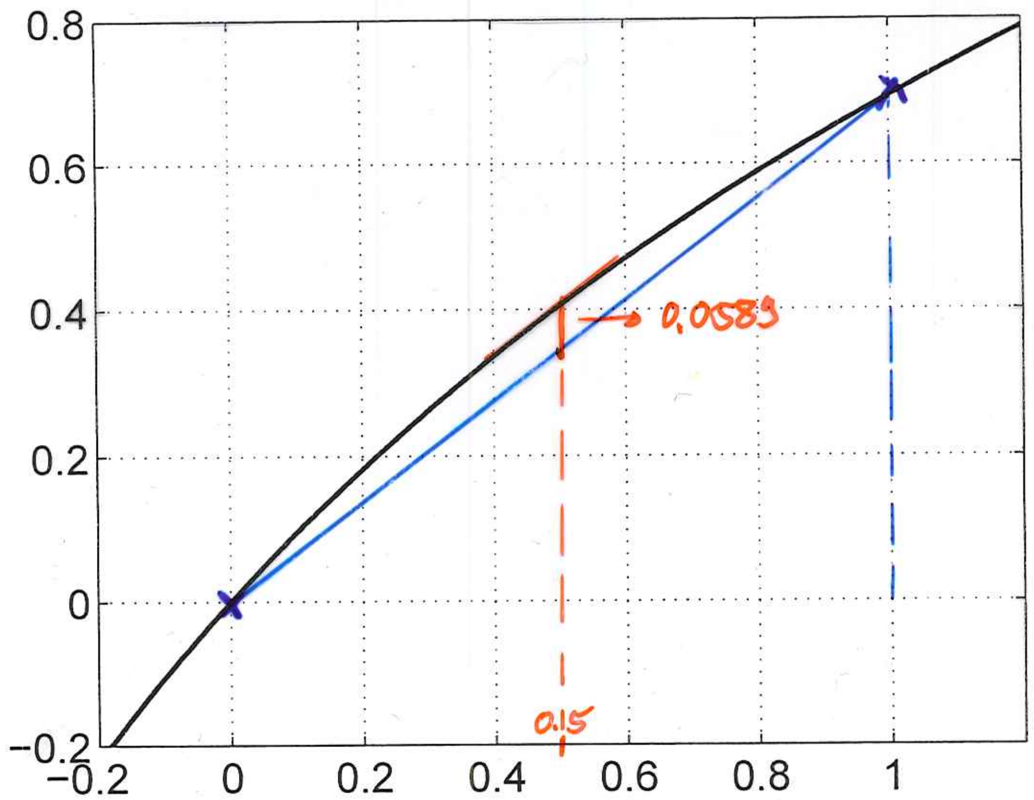
$$f(x) - P(f \mid x_0, x_1, x_2)(x) = (x-x_0)(x-x_1)(x-x_2)\,\frac{f'''(\xi_2)}{6}$$

mit $\xi_2 \in [x_0, x_2]$

$$\Rightarrow \left| f(x) - P(f \mid x_0, x_1, x_2)(x) \right| \leq \underbrace{\max_{x\in[x_0,x_2]} \left|(x-x_0)(x-x_1)(x-x_2)\right|}_{\substack{\text{abh. von Stützstellen} \\ \omega(x)}} \underbrace{\max_{\xi_2\in[x_0,x_2]} \left|\frac{f'''(\xi_2)}{6}\right|}_{\substack{\text{abh. von } f(x) \\ M(x)}}$$

Beispiel: 8.24:

$$f(x) = \log(1+x)$$



$x_0 = 0$ $\qquad$ $x_1 = 1$

Ableitungen: $f'(x) = \dfrac{1}{1+x}$

$$f''(x) = -\frac{1}{(1+x)^2}$$

$$f'''(x) = \frac{2}{(1+x)^3}$$

Lineare Interpolation:

$$\max_{x\in[0,1]} |x(x-1)| \Rightarrow x^* = 0.5 \Rightarrow \max_{x\in[0,1]} |x^*(x^*-1)| = \frac{1}{4}$$

$$\max_{\xi\in[0,1]} \left|\frac{f''(\xi)}{2}\right| = \frac{1}{2} \quad \Rightarrow |f(x) - P(f|0,1)(x)| \leq \frac{1}{8}$$

N 8.8